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New Perspectives On Product Constructions of Einstein Fuzzy Graphs

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Abstract

This paper introduces Einstein fuzzy graphs as a novel framework for representing and analyzing uncertain or imprecise relationships between entities using fuzzy set theory. By employing the Einstein t-norm and t-conorm operations, fuzzy analogs of fundamental graph concepts are developed, emphasizing both their theoretical depth and practical relevance. A key contribution of this work is the introduction of modified product operations, which are carefully designed to ensure that the product of any two Einstein fuzzy graphs results in a valid Einstein fuzzy graph. These modified operations are shown to be more suitable than traditional product operations when dealing with the nuanced behavior of fuzzy relationships. Definitions for various types of graph products are presented, along with illustrative examples. The properties of these products are thoroughly analyzed and comparisons with traditional operations underscore the advantages of the modified approach. Also, key propositions and theorems are established to support a comprehensive understanding of Einstein fuzzy graphs and their structure. Real-life applications are also discussed, demonstrating the practical utility of this framework.

Keywords: fuzzy sets; Einstein t-norm; Einstein t-conorm; Einstein fuzzy graph.

1 Introduction

Fuzzy sets were motivated by the need to model uncertainty and vagueness inherent in many real-world situations where traditional binary logic falls short. Unlike classical sets that classify elements as either belonging or not belonging, fuzzy sets allow partial membership, reflecting degrees of truth. The motivation also stems from human reasoning, which often involves imprecise information and subjective judgments. By capturing this fuzziness, fuzzy sets enable more natural and effective decision-making in complex systems. Yusoff et al. [22] introduced circular $q-$rung orthopair fuzzy sets to overcome the limitations of the intuitionistic fuzzy interpretation triangle, enabling broader representation of uncertainty along with enhanced algebraic and modal operations.

Graphs visually depict the intricate web of connections and interactions between entities, serving as a powerful tool for analyzing and understanding complex relationships with ease. Objects are represented as nodes, while their connections or interactions are illustrated as edges. When the relationships or attributes of objects are uncertain or vague, the graph is termed a fuzzy graph. Susanti [21] provided a structural description of the 2-token graph derived from the disjoint union of multiple graphs, complementing existing studies on the properties and behavior of k-token graphs. Fuzzy relations are widely and critically applied in various fields, including decision-making, cluster analysis, computer networks, pattern recognition, neural networks and expert systems. In each case, the fundamental mathematical structure is a fuzzy graph. The foundation of modern uncertainty theory was laid by Zadeh [23] in his seminal 1965 paper.

Fuzzy graphs were motivated by the need to represent relationships and connections that are not simply present or absent, but can have varying degrees of strength or uncertainty. Traditional graph theory assumes crisp edges and vertices either they exist or they don't limiting its use in real world scenarios where interactions are often vague or partial. By introducing fuzziness into graphs, it becomes possible to model complex systems such as social networks, communication systems, or biological interactions, where relationships can be uncertain or imprecise. This allows for more realistic and flexible analysis of networks where edges and nodes have degrees of membership, reflecting real-life ambiguity. Ultimately, fuzzy graphs extend classical graph theory to better handle uncertainty, imprecision and gradual transitions in connectivity.

Graph products are motivated by the need to combine multiple graphs into one structure to study complex systems with interacting components, preserving their individual properties. They help model multi dimensional networks like transportation or communication systems. Fuzzy graph products extend this idea to networks with uncertain or partial connections, allowing the combination of fuzzy graphs to better represent real-world systems where relationships aren't simply on or off but have varying strengths.

Researchers have shown significant interest in fuzzy sets and fuzzy graphs over the past several decades. Currently, ongoing research continues to explore the applications of fuzzy sets, with particular emphasis on fuzzy graphs. However, the concept of intuitionistic Einstein fuzzy graphs remains an unexplored area in the field. Menger [13] explored t-norms and t-conorms within the context of probabilistic metric spaces. Numerous researchers have proposed various types of T-operators (t-norms and t-conorms) for the intersection and union of fuzzy sets. The classical t-operators, $\min$ and $\max$, developed by Zadeh, are widely applied in fuzzy logic, particularly in decision-making processes and the study of fuzzy graph theory.

It is important to note that, both experimentally and theoretically, t-operators other than $\min$ and $\max$ often yield better results in specific contexts, especially in decision-making scenarios.

The selection of suitable t-operators for a particular application involves considering their inherent properties, simplicity, compatibility with the model's requirements and their effectiveness in both hardware and software environments.

As research on these operators has progressed, a broader range of t-operator options has emerged, offering improved adaptability and performance for specific investigations. Sunitha and Vijayakumar [20] developed new operations for fuzzy graphs and redefined the concept of complement to align with its standard interpretation in crisp graph theory. Mordeson and Chang-Shyh [14] introduced the concept of strong fuzzy graphs and investigated various operations associated with fuzzy graphs. Al-Hawary [2] introduced the notions of parallel and series connections within the framework of balanced fuzzy graphs. The concept of strong arcs in fuzzy graphs was proposed by Bhutani and Rosenfeld [10].

The notions of eccentricity and center in fuzzy graphs were first introduced by Bhattacharya [7]. Mordeson and Chang-Shyh [14] further explored operations on fuzzy subgraphs, including Cartesian product, composition, union and join. Additionally, Bhutani and Battou [9] studied the preservation of the M-strong property in fuzzy graphs. The investigation of three novel product operations in fuzzy product graphs was conducted by Al-Hawary and Horani [3], significantly advancing the study of fuzzy graphs through innovative approaches to product structures. Nagoorgani and Malarvizhi [15] examined the properties of complements in fuzzy graphs, while S. Ashraf et al. [4] introduced the concept of Dombi fuzzy graphs. Furthermore, Bhutani [8] proposed that a fuzzy group could be embedded within the automorphism group of a fuzzy graph.

Klement et al. [12] and co-authors explored the core analytical and algebraic properties of triangular norms. Similarly, Bera et al. [6] studied vertex connectivity in intuitionistic fuzzy graphs. The work by Raja et al. [16] introduced the notion of a hesitant bipolar-valued fuzzy graph (HBVFG), a framework that captures both positive and negative viewpoints. Zuo et al. [24] defined the concept of a picture fuzzy graph, which is based on picture fuzzy relations. Atanassov [5] proposed the concepts of intuitionistic fuzzy relations and intuitionistic fuzzy graphs, extending traditional fuzzy set theory to more effectively model uncertainty and ambiguity in relational and graphical data.

Romdhini et al. [17] introduced the concept of the signless Laplacian matrix for interval-valued fuzzy directed graphs and provided its formal definition. They investigated important spectral properties of this matrix, including eigenvalues, spectrum, spectral radius and graph energy. These results contribute significantly to the understanding and analysis of interval-valued fuzzy directed graphs under uncertainty. Shi et al. [19] investigated various types of energy, including Laplacian and skew-Laplacian energy, in both picture fuzzy graphs and picture fuzzy digraphs and discussed several of their fundamental properties.

In a study conducted by Shao et al. [18] the authors introduced and examined several new operations on vague graphs. These operations include rejection, maximal product, symmetric difference and residue product. The research aims to present the key properties of these operations and explore their mathematical foundations and applications within the domain of vague graph theory. This work contributes to the growing body of literature on fuzzy and vague structures, offering insights that may enhance their utility in complex network analysis and decision-making models.

This paper proposes modified definitions for the Cartesian product, Semi-strong product, Symmetric composition and Composition of two graphs within the intuitionistic Einstein fuzzy setting [11]. It also builds upon the work of Jacob et al. [11] on Einstein fuzzy graphs, offering new perspectives and extending the scope of their original findings. In addition, this paper discusses the Quasi-I Cartesian product and Quasi-II Cartesian product, along with their modified forms to satisfy the Einstein property. The singular M-Lexicographic product is also examined, with illustrative examples provided. Also, properties of these products are explored in detail. Applications of these products in the medical field are also discussed, particularly using the Cartesian and M-Cartesian products.

2 Preliminaries

A graph $\mathfrak{G}_G$=$(\mathfrak{V}_V, \mathfrak{E}_E)$ is a mathematical form or structure containing a set of vertices $\mathfrak{V}_V = \mathfrak{V}_V(\mathfrak{G}_G)$ and a set of edges $\mathfrak{E}_E = \mathfrak{E}_E(\mathfrak{G}_G)$, where every edge is an unordered pair of vertices [4]. If a vertex $\mathfrak{y}_y$ is joined by a node or vertex $\mathfrak{x}_x$, then $\mathfrak{y}_y$ is called the neighbor of $\mathfrak{x}_x$. The count of edges joined with a vertex $\mathfrak{x}_x$ of a graph $\mathfrak{G}_G$ is called the degree of $\mathfrak{x}_x$ in $\mathfrak{G}_G$. The degree of $\mathfrak{x}_x$ in $\mathfrak{G}_G$ is denoted by $d_{\mathfrak{G}_G}(\mathfrak{x}_x)$. In light of the previous works, this study introduces new definitions and derives several interesting results.

Let $\mathfrak{G}_{G1} = (\mathfrak{V}_{V1}, \mathfrak{E}_{E1})$ and $\mathfrak{G}_{G2} = (\mathfrak{V}_{V2}, \mathfrak{E}_{E2})$ be two graphs. The standard product of these two graphs is defined as follows.

Assume that $(\mathfrak{u}_{x1}, \mathfrak{u}_{x2})$,$(\upsilon_{y1}, \upsilon_{y2}) \in \mathfrak{V}_{G1} \times \mathfrak{V}_{G2}$, where $\mathfrak{V}_{G1} \times \mathfrak{V}_{G2}$ is the vertex set of each product graph.

- The direct product or Tensor product [4],

$$\mathfrak{E}_E(\mathfrak{G}_{G1} \times \mathfrak{G}_{G2}) = \Big\{\big((\mathfrak{u}_{x1}, \mathfrak{u}_{x2}), (\upsilon_{y1}, \upsilon_{y2})\big)|(\mathfrak{u}_{x1}, \upsilon_{y1}) \in \mathfrak{E}_{E1} \quad \text{and} \quad (\mathfrak{u}_{x2}, \upsilon_{y2}) \in \mathfrak{E}_{E2}\Big\}.$$

- The Cartesian product [4],

$$\mathfrak{E}_E(\mathfrak{G}_{G1} \Box \mathfrak{G}_{G2}) = \Big\{\big((\mathfrak{u}_{x1}, \mathfrak{u}_{x2}), (\upsilon_{y1}, \upsilon_{y2})\big)|\mathfrak{u}_{x1} = \upsilon_{y1}, \quad \text{and} \quad (\mathfrak{u}_{x2}, \upsilon_{y2}) \in \mathfrak{E}_{E2}, \quad \text{or} \quad (\mathfrak{u}_{x1}, \upsilon_{y1}) \in \mathfrak{E}_{E1}, \quad \text{and} \quad \mathfrak{u}_{x2} = \upsilon_{y2}\Big\}.$$

- The semi-strong product [4],

$$\mathfrak{E}_E(\mathfrak{G}_{G1} \bullet \mathfrak{G}_{G2}) = \Big\{\big((\mathfrak{u}_{x1}, \mathfrak{u}_{x2}), (\upsilon_{y1}, \upsilon_{y2})\big)|\mathfrak{u}_{x1} = \upsilon_{y1}, \quad \text{and} \quad (\mathfrak{u}_{x2}, \upsilon_{y2}) \in \mathfrak{E}_{E2}, \quad \text{or} \quad (\mathfrak{u}_{x1}, \upsilon_{y1}) \in \mathfrak{E}_{E1}, \quad \text{and} \quad (\mathfrak{u}_{x2}, \upsilon_{y2}) \in \mathfrak{E}_{E2}\Big\}.$$

- The strong product or symmetric composition [4],

$$\mathfrak{E}_E(\mathfrak{G}_{G1} \boxtimes \mathfrak{G}_{G2}) = \mathfrak{E}_E(\mathfrak{G}_{G1} \Box \mathfrak{G}_{G2}) \cup \mathfrak{E}_E(\mathfrak{G}_{G1} \times \mathfrak{G}_{G2}).$$

- The Lexicographic product or composition [4],

$$\mathfrak{E}_E(\mathfrak{G}_{G1}[\mathfrak{G}_{G2}]) = \Big\{\big((\mathfrak{u}_{x1}, \mathfrak{u}_{x2})(\upsilon_{y1}, \upsilon_{y2})\big)|(\mathfrak{u}_{x1}, \upsilon_{y1}) \in \mathfrak{E}_{E1}, \quad \text{or} \quad \mathfrak{u}_{x1} = \upsilon_{y1}, \quad \text{and} \quad (\mathfrak{u}_{x2}, \upsilon_{y2}) \in \mathfrak{E}_{E2}.$$

Definition 2.1. [11] *A fuzzy subset ψ of a set $\mathfrak{V}_V$ is a function,*

$$\psi : \mathfrak{V}_V \to [0, 1].$$

A fuzzy relation [4] on $\mathfrak{V}_V$ is a mapping,

$$\xi : \mathfrak{V}_V \times \mathfrak{V}_V \to [0, 1].$$

Definition 2.2. [4] *A binary operation* $\mathfrak{T}_T : [0,1]^2 \to [0,1]$ *is called a t-norm if it satisfies the following properties for all for all* $\mathbbm{u}_x, \mathbbm{v}_y, \mathbbm{w}_z \in [0,1]$*:*

1. *Boundary condition:* $\mathfrak{T}_T(1, \mathbbm{u}_x) = \mathbbm{u}_x$.
2. *Commutativity:* $\mathfrak{T}_T(\mathbbm{u}_x, \mathbbm{v}_y) = \mathfrak{T}_T(\mathbbm{v}_y, \mathbbm{u}_x)$.
3. *Associativity:* $\mathfrak{T}_T(\mathbbm{u}_x, \mathfrak{T}_T(\mathbbm{v}_y, \mathbbm{w}_z) = \mathfrak{T}_T(\mathfrak{T}_T(\mathbbm{u}_x, \mathbbm{v}_y), \mathbbm{w}_z)$.
4. *Monotonicity:* $\mathfrak{T}_T(\mathbbm{u}_x, \mathbbm{v}_y) \leq \mathfrak{T}_T(\mathbbm{v}_y, \mathbbm{w}_z)$ *if* $\mathbbm{v}_x \leq \mathbbm{w}_z$.

Definition 2.3. [4] *A binary operation* $\mathfrak{T}_T$*:*$[0,1]^2 \to [0,1]$ *is cosidering a triangular conorm (t-conorm) if and only if there exists a t-norm* $\mathfrak{T}_T$ *such that for every pair,*

$$(\mathbbm{u}_x, \mathbbm{v}_y) \in [0,1]^2 \mathfrak{S}_S(\mathbbm{u}_x, \mathbbm{v}_y) = 1 - \mathfrak{T}_T(1 - \mathbbm{u}x, 1 - \mathbbm{v}_y),$$

holds.

Consider the Hamacher family of t-norms and t-conorms,

$$\frac{\mathbbm{u}_x \mathbbm{v}_y}{\lambda + (1-\lambda)(\mathbbm{u}_x + \mathbbm{v}_y - \mathbbm{u}_x \mathbbm{v}_y)}, \qquad \lambda > 0, \qquad (\textit{Hamacher t-norm})$$

$$\frac{\mathbbm{u}_x + \mathbbm{v}_y + (\lambda - 2)\mathbbm{u}_x \mathbbm{v}_y}{1 + (\lambda - 1)\mathbbm{u}_x \mathbbm{v}_y}, \qquad \lambda > 0, \qquad (\textit{Hamacher t-conorm})$$

An example of $\mathfrak{T}_T$*-operators is*

$$\mathfrak{T}_T(\mathbbm{u}_x, \mathbbm{v}_y) = \frac{\mathbbm{u}_x \mathbbm{v}_y}{1 + (1 - \mathbbm{u}_x)(1 - \mathbbm{v}_y)}, \qquad (\textit{Einstein t-norm})$$

$$\mathfrak{S}_S(\mathbbm{u}_x, \mathbbm{v}_y) = \frac{\mathbbm{u}_x + \mathbbm{v}_y}{1 + \mathbbm{u}_x \mathbbm{v}_y}, \qquad (\textit{Einstein t-conorm})$$

These operators are obtained by putting $\lambda = 2$ *in the Hamacher family. Also,*

$$\max(\mathbbm{u}_x + \mathbbm{v}_y - 1, 0) \leq \frac{\mathbbm{u}_x \mathbbm{v}_y}{1 + (1 - \mathbbm{u}_x)(1 - \mathbbm{v}_y)} \leq \frac{\mathbbm{u}_x \mathbbm{v}_y}{\mathbbm{u}_x + \mathbbm{v}_y - \mathbbm{u}_x \mathbbm{v}_y} \leq \min(\mathbbm{u}_x, \mathbbm{v}_y),$$

and

$$(\mathbbm{u}_x + \mathbbm{v}_y - \mathbbm{u}_x \mathbbm{v}_y) \leq \frac{\mathbbm{u}_x + \mathbbm{v}_y - 2\mathbbm{u}_x \mathbbm{v}_y}{1 - \mathbbm{u}_x \mathbbm{v}_y} \leq \max(\mathbbm{u}_x, \mathbbm{v}_y) \leq \frac{\mathbbm{u}_x + \mathbbm{v}_y}{1 + \mathbbm{u}_x \mathbbm{v}_y} \leq \min(\mathbbm{u}_x + \mathbbm{v}_y, 1).$$

Definition 2.4. [11] *An intuitionistic fuzzy set* $\mathfrak{A}_I$ *on the set* $\mathfrak{X}_I$ *is represented by a pair of mappings* $\mu_{\mathfrak{A}_I} : \mathfrak{X}_I \to [0,1]$ *which is called the membership function and* $\nu_{\mathfrak{A}_I} : \mathfrak{X}_I \to [0,1]$ *which is referred to as the non-membership function. It is typically denoted as* $\mathfrak{A}_I = (\mathfrak{X}_I, \mu_{\mathfrak{A}_I}, \nu_{\mathfrak{A}_I})$.

Definition 2.5. [1] *A fuzzy graph, defined on a vertex set* $\mathfrak{V}_V$*, is represented as a pair* $\mathcal{G}_G = (\sigma_{\mathcal{G}}, \mu_{\mathcal{G}})$*, where* $\sigma_{\mathcal{G}} : \mathfrak{V}_V \to [0;1]$ *assigns a membership value to each vertex and* $\mu_{\mathcal{G}} : \mathfrak{V}_V \times \mathfrak{V}_V \to [0;1]$ *defines a fuzzy relation between vertex pairs. This relation must satisfy the condition* $\mu_{\mathcal{G}}(\mathbbm{x}_x, \mathbbm{y}_y) \leq \sigma_{\mathcal{G}}(\mathbbm{x}_x) \wedge \sigma_{\mathcal{G}}(\mathbbm{y}_y)$ *for all* $\mathbbm{x}_x, \mathbbm{y}_y \in \mathfrak{V}_V$*, where* $\wedge$ *stands for minimum.*

Definition 2.6. [11] *An Einstein fuzzy graph with a finite set* $\mathfrak{V}_V$ *as the underlying set is a pair* $\mathcal{G}_G = (\psi, \xi)$*, where* $\psi : \mathfrak{V}_V \to [0,1]$ *is a fuzzy subset in* $\mathfrak{V}_V$ *and* $\xi : \mathfrak{V}_V \times \mathfrak{V}_V \to [0,1]$ *is a symmetric fuzzy relation on* ψ *such that,*

$$\xi(\mathbbm{u}_x, \mathbbm{v}_y) \leq \frac{\psi(\mathbbm{u}_x)\psi(\mathbbm{v}_y)}{1 + \Big(1 - \psi(\mathbbm{u}_x)\Big)\Big(1 - \psi(\mathbbm{v}_y)\Big)} = \frac{\psi(\mathbbm{u}_x)\psi(\mathbbm{v}_y)}{2 - \Big(\psi(\mathbbm{u}_x) + \psi(\mathbbm{v}_y)\Big) + \psi(\mathbbm{u}_x)\psi(\mathbbm{v}_y)},$$

for all $\mathbbm{u}_x, \mathbbm{v}_y \in \mathfrak{V}_V$*. We designate* ψ *as the Einstein fuzzy vertex set of* $\mathcal{G}_G$ *and* ξ *as its Einstein fuzzy edge set.*

Definition 2.7. [11] *An Einstein fuzzy edge graph $\mathcal{G}_G$ is defined as an Einstein fuzzy graph of a graph $\mathfrak{G}_G$ in which every vertex is crisply included, while each edge is assigned a fuzzy membership degree from the interval $[0, 1]$.*

Definition 2.8. [11] *Let ψ_{V_i} be a fuzzy subset of $\mathfrak{V}_{Vi}$ and ξ_{E_i} be a fuzzy subset of $\mathfrak{E}_{Ei}$, $i = 1, 2$. Define the direct product of the Einstein fuzzy graphs,*

$$\mathcal{G}_{G1} = (\psi_{V1}, \xi_{E1}) \text{ of } \mathfrak{G}_{G1} = (\mathfrak{V}_{V1}, \mathfrak{E}_{E1}), \quad \text{and} \quad \mathcal{G}_{G2} = (\psi_{V2}, \xi_{E2}) \text{ of } \mathfrak{G}_{G2} = (\mathfrak{V}_{V2}, \mathfrak{E}_{E2}),$$

as

$$\mathcal{G}_{G1} \times \mathcal{G}_{G2} = (\psi_{V1} \times \psi_{V2}, \xi_{E1} \times \xi_{E2}),$$

where the components are defined as follows:

- *for all $(\mathfrak{u}_{x1}, \mathfrak{u}_{x2}) \in \mathfrak{V}_{V1} \times \mathfrak{V}_{V2}$,*

$$(\psi_{V1} \times \psi_{V2})(\mathfrak{u}_{x1}, \mathfrak{u}_{x2}) = \frac{\psi_{V1}(\mathfrak{u}_{x1})\psi_{V2}(\mathfrak{u}_{x2})}{1 + \Big(1 - \psi_{V1}(\mathfrak{u}_{x1})\Big)\Big(1 - \psi_{V2}(\mathfrak{u}_{x2})\Big)}.$$

- *for all $(\mathfrak{u}_{x1}, \mathfrak{v}_{y1}) \in \mathfrak{E}_{E1}$ and $(\mathfrak{u}_{x2}, \mathfrak{v}_{y2}) \in \mathfrak{E}_{E2}$,*

$$(\xi_{E1} \times \xi_{E2})\Big((\mathfrak{u}_{x1}, \mathfrak{u}_{x2}), (\mathfrak{v}_{y1}, \mathfrak{v}_{y2})\Big) = \frac{\xi_{E1}(\mathfrak{u}_{x1}, \mathfrak{v}_{y1})\xi_{E2}(\mathfrak{u}_{x2}, \mathfrak{v}_{y2})}{1 + \Big(1 - \xi_{E1}(\mathfrak{u}_{x1}, \mathfrak{v}_{y1})\Big)\Big(1 - \xi_{E2}(\mathfrak{u}_{x2}, \mathfrak{v}_{y2})\Big)}.$$

Definition 2.9. [1] *An intuitionistic fuzzy graph is represented as,*

$$\mathcal{G}_G = (\mathfrak{V}_V, \mathfrak{E}_E, \sigma_{\mathcal{G}}, \mu_{\mathcal{G}}),$$

where

$$\sigma_{\mathcal{G}} = (\sigma_{\mathcal{G}1}, \sigma_{\mathcal{G}2}), \quad \text{and} \quad \mu_{\mathcal{G}} = (\mu_{\mathcal{G}1}, \mu_{\mathcal{G}2}),$$

denote fuzzy sets. The function $\sigma_{\mathcal{G}i} : \mathfrak{V}_V \to [0, 1]$ and $\mu_{\mathcal{G}i} : \mathfrak{V}_V \times \mathfrak{V}_V \to [0; 1]$. Again $\mu_{\mathcal{G}i}$ defines a fuzzy relationship on $\sigma_{\mathcal{G}}$ such that for any two vertices $\mathfrak{x}_x, \mathfrak{y}_y \in \mathfrak{V}_V$, the condition $\mu_{\mathcal{G}1}(\mathfrak{x}_x, \mathfrak{y}_y) \leq \sigma_{\mathcal{G}1}(\mathfrak{x}_x) \wedge \sigma_{\mathcal{G}1}(\mathfrak{y}_y)$ and $\mu_{\mathcal{G}2}(\mathfrak{x}_x, \mathfrak{y}_y) \leq \sigma_{\mathcal{G}2}(\mathfrak{x}_x) \vee \sigma_{\mathcal{G}2}(\mathfrak{y}_y)$ holds, where $\wedge$ and $\vee$ represents the minimum and maximum operations, $i \in \{1, 2\}$.

In this paper, $\mathfrak{G}_G$ refers to the crisp graph, whereas $\mathcal{G}_G$ denotes the Einstein fuzzy graph.

3 Modified Products of Einstein Fuzzy Graphs

In this part, we revise the definition of the "Cartesian", "Semi-strong", "Strong" and "Lexicographic" products of fuzzy graphs to guarantee that the resulting graph satisfies the properties of an Einstein fuzzy graph. This involves applying specific transformations or adjustments to the membership functions in these products to reflect the unique characteristics of Einstein fuzzy graphs. These modifications will ensure that the resultant graph adheres to the properties and constraints that define Einstein fuzzy graphs.

Definition 3.1. *Let ψ_{Vi} be a subset of $\mathfrak{V}_i$ and let ξ_{Ei} be a subset of $\mathfrak{E}_i$, $i = 1, 2$. Define the Quasi-I Cartesian product $\mathcal{G}_{G1} \sqsubset_Q \mathcal{G}_{G2} = (\psi_{V1} \sqsubset_Q \psi_{V2}, \xi_{E1} \sqsubset_Q \xi_{E2})$ of the Einstein fuzzy graphs $\mathcal{G}_{G1} = (\psi_{V1}, \xi_{E1})$ of $\mathfrak{G}_{G1} = (\mathfrak{V}_{V1}, \mathfrak{E}_{E1})$ and $\mathcal{G}_{G2} = (\psi_{V2}, \xi_{E2})$ of $\mathfrak{G}_{G2} = (\mathfrak{V}_{V2}, \mathfrak{E}_{E2})$, respectively as follows:*

i) $(\psi_{V1} \sqsubset_Q \psi_{V2})(\mathfrak{u}_{x1}, \mathfrak{u}_{x2}) = \dfrac{\psi_{V1}(\mathfrak{u}_{x1})\psi_{V2}(\mathfrak{u}_{x2})}{1 + \Big(1 - \psi_{V1}(\mathfrak{u}_{x1})\Big)\Big(1 - \psi_{V2}(\mathfrak{u}_{x2})\Big)}$,
for all $(\mathfrak{u}_{x1}, \mathfrak{u}_{x2}) \in \mathfrak{V}_{V1} \times \mathfrak{V}_{V2}$.

ii) $(\xi_{E1} \sqsubset_Q \xi_{E2})\Big((\mathfrak{u}_x, \mathfrak{u}_{x2})(\mathfrak{u}_x, \mathfrak{v}_{y2})\Big) = \dfrac{\psi_{V1}(\mathfrak{u}_x)\xi_{E2}(\mathfrak{u}_{x2}, \mathfrak{v}_{y2})}{1 + \Big(1 - \psi_{V1}(\mathfrak{u}_{\mathfrak{x}})\Big)\Big(1 - \xi_{E2}(\mathfrak{u}_{x2}, \mathfrak{v}_{y2})\Big)}$,
for all $\mathfrak{u}_x \in \mathfrak{V}_{y1}$, *for all* $(\mathfrak{u}_{x2}, \mathfrak{v}_{y2}) \in \mathfrak{E}_{E2}$.

Definition 3.2. *Let ψ_{Vi} be a subset of $\mathfrak{V}_i$ and let ξ_{Ei} be a subset of $\mathfrak{E}_i$, $i = 1, 2$. Define the Quasi-II Cartesian product $\mathcal{G}_{G1} \sqsupset_Q \mathcal{G}_{G2} = (\psi_{V1} \sqsupset_Q \psi_{V2}, \xi_{E1} \sqsupset_Q \xi_{E2})$ of the Einstein fuzzy graphs,*

$$\mathcal{G}_{G1} = (\psi_{V1}, \xi_{E1}) \text{ of } \mathfrak{G}_{G1} = (\mathfrak{V}_{V1}, \mathfrak{E}_{E1}) \quad \text{and} \quad \mathcal{G}_{G2} = (\psi_{V2}, \xi_{E2}) \text{ of } \mathfrak{G}_{G2} = (\mathfrak{V}_{V2}, \mathfrak{E}_{E2}),$$

respectively as follows:

i) $(\psi_{V1} \sqsupset_Q \psi_{V2})(\mathfrak{u}_{x1}, \mathfrak{u}_{x2}) = \dfrac{\psi_{V1}(\mathfrak{u}_{x1})\psi_{V2}(\mathfrak{u}_{x2})}{1 + \Big(1 - \psi_{V1}(\mathfrak{u}_{x1})\Big)\Big(1 - \psi_{V2}(\mathfrak{u}_{x2})\Big)}$,
for all $(\mathfrak{u}_{x1}, \mathfrak{u}_{x2}) \in \mathfrak{V}_{V1} \times \mathfrak{V}_{V2}$.

ii) $(\xi_{E1} \sqsupset_Q \xi_{E2})\Big((\mathfrak{u}_{x1}, \mathfrak{w}_y)(\mathfrak{v}_{y1}, \mathfrak{w}_y)\Big) = \dfrac{\xi_{E1}(\mathfrak{u}_{x1}, \mathfrak{v}_{y1})\psi_{V2}(\mathfrak{w}_y)}{1 + \Big(1 - \xi_{E1}(\mathfrak{u}_{x1}, \mathfrak{v}_{y1})\Big)\Big(1 - \psi_{V2}(\mathfrak{w}_y)\Big)}$,
for all $(\mathfrak{u}_{x1}, \mathfrak{v}_{y1}) \in \mathfrak{E}_{E1}$, *for all* $\mathfrak{w}_y \in \mathfrak{V}_{V2}$.

Remark 3.1. *The Definitions 3.1 and 3.2 together form the Cartesian product of Einstein fuzzy graphs [11].*

Example 3.1. *Consider two Einstein fuzzy graphs $\mathcal{G}_{G1}$ and $\mathcal{G}_{G2}$ as in Figures 1, 2, 3 and 4.*

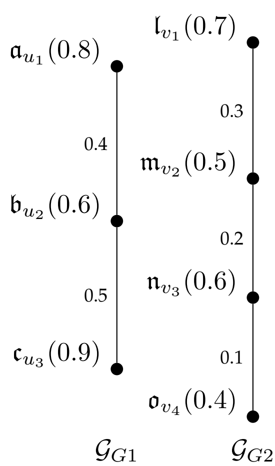

Figure 1: Einstein fuzzy graphs.

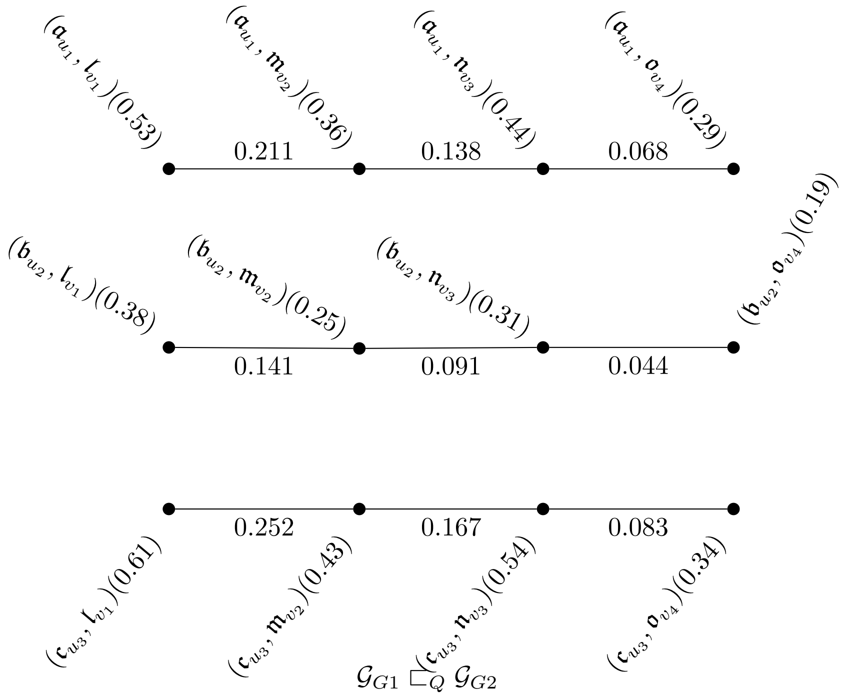

Figure 2: Fuzzy graph: $\mathcal{G}_{G1} \sqsubset_Q \mathcal{G}_{G2}$.

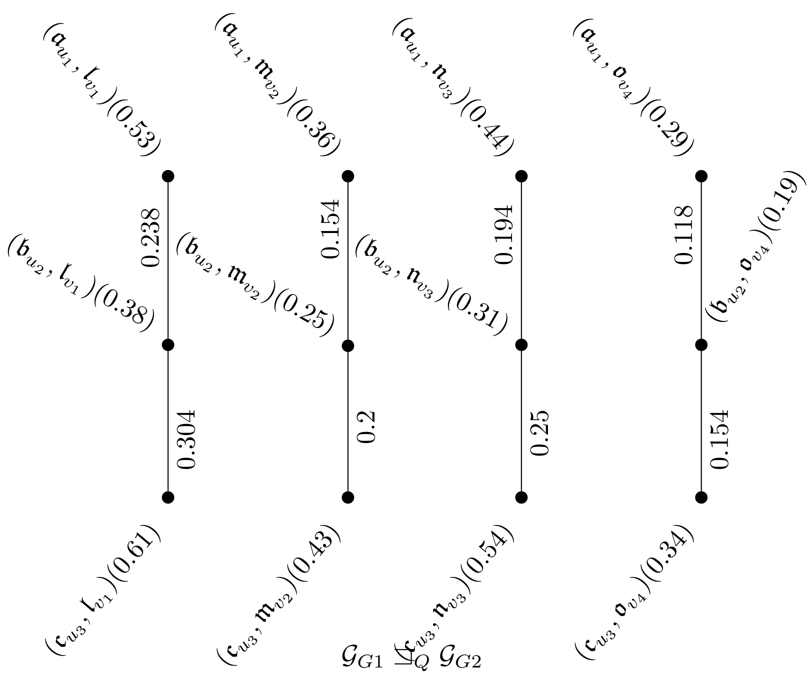

Figure 3: Fuzzy graph: $\mathcal{G}_{G1} \sqsupset_Q \mathcal{G}_{G2}$.

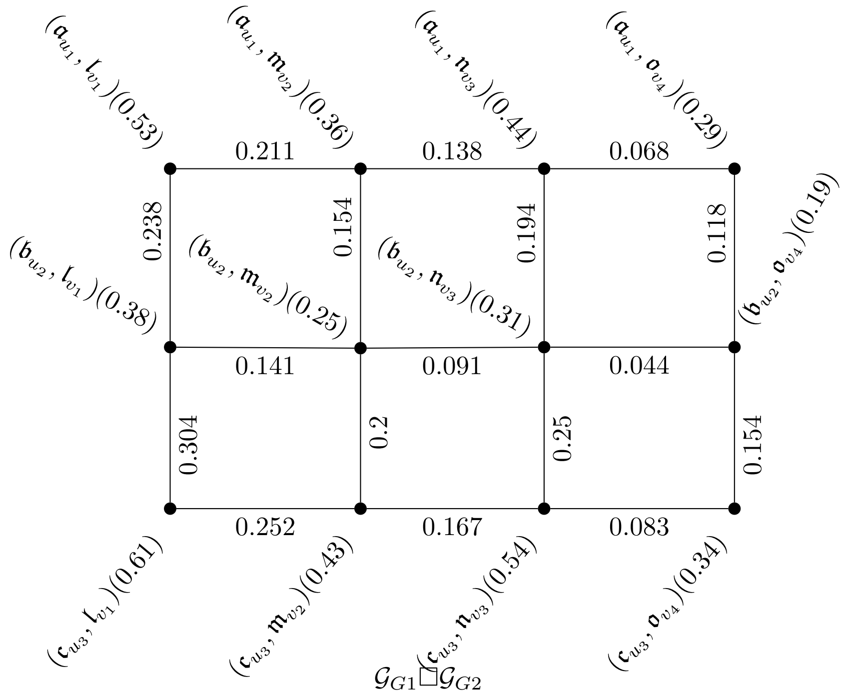

Figure 4: Fuzzy graph: $\mathcal{G}_{G1}\Box\mathcal{G}_{G2}$.

We can easily say that,

$$(\xi_{E1} \sqsubset_Q \xi_{E2})\Big[(b_{u_2}, l_{v_1})(b_{u_2}, m_{v_2})\Big] = 0.141 \nleq 0.065 = \frac{0.38 \times 0.25}{1 + \Big[(1-0.38)(1-0.25)\Big]}$$

$$= \frac{(\psi_{V1} \sqsubset_Q \psi_{V2})(b_{u_2}, l_{v_1}) \cdot (\psi_{V1} \sqsubset_Q \psi_{V2})(b_{u_2}, m_{v_2})}{1 + \Big[1 - (\psi_{V1} \sqsubset_Q \psi_{V2})(b_{u_2}, l_{v_1})\Big]\Big[1 - (\psi_{V1} \sqsubset_Q \psi_{V2})(b_{u_2}, m_{v_2})\Big]},$$

$$(\xi_{E1} \sqsupset_Q \xi_{E2})\Big[(b_{u_2}, o_{V_4})(c_{u_3}, o_{v_4})\Big] = 0.154 \nleq 0.0421 = \frac{0.19 \times 0.34}{1 + \Big[(1-0.19)(1-0.34)\Big]}$$

$$= \frac{(\psi_{V1} \sqsupset_Q \psi_{V2})(b_{u_2}, o_{v_4}) \cdot (\psi_{V1} \sqsupset_Q \psi_{V2})(c_{u_3}, o_{V_4})}{1 + \Big[1 - (\psi_{V1} \sqsupset_Q \psi_{V2})(b_{u_2}, o_{v_4})\Big]\Big[1 - (\psi_{V1} \sqsupset_Q \psi_{V2})(c_{u_3}, o_{v_4})\Big]}.$$

Hence, $\mathcal{G}_{G1} \sqsubset_Q \mathcal{G}_{G2}$ and $\mathcal{G}_{G1} \sqsupset_Q \mathcal{G}_{G2}$ are not Einstein fuzzy graphs. Therefore, we can conclude that $\mathcal{G}_{G1}\Box\mathcal{G}_{G2}$ is not an Einstein fuzzy graph [11].

Definition 3.3. *The* ***"M-Cartesian product"*** $\mathcal{G}_{G1}\Box_M\mathcal{G}_{G2} = (\psi_{V1}\Box_M\psi_{V2}, \xi_{E1}\Box_M\xi_{E2})$ *of the Einstein fuzzy graphs,*

$$\mathcal{G}_{G1} = (\psi_{V1}, \xi_{E1}) \ \ of \ \ \mathfrak{G}_{G1} = (\mathfrak{V}_{V1}, \mathfrak{E}_{E1}), \quad and \quad \mathcal{G}_{G2} = (\psi_{V2}, \xi_{E2}) \ \ of \ \ \mathfrak{G}_{G2} = (\mathfrak{V}_{V2}, \mathfrak{E}_{E2}),$$

is defined as follows. Here, ψ_{V_i} *and* ξ_{E_i} *are the fuzzy subsets of* $\mathfrak{V}_{V_i}$ *and* $\mathfrak{E}_{E_i}$*,* $i \in \{1, 2\}$*:*

i) $(\psi_{V1}\Box_M\psi_{V2})(\mathfrak{u}_{x1},\mathfrak{u}_{x2}) = \dfrac{\psi_{V1}(\mathfrak{u}_{x1})\psi_{V2}(\mathfrak{u}_{x2})}{1+\Big(1-\psi_{V1}(\mathfrak{u}_{x1})\Big)\Big(1-\psi_{V2}(\mathfrak{u}_{x2})\Big)},$
for all $(\mathfrak{u}_{x1},\mathfrak{u}_{x2}) \in \mathfrak{V}_{V1}\times\mathfrak{V}_{V2}$.

ii) $(\xi_{E1}\Box_M\xi_{E2})(\mathfrak{u}_x,\mathfrak{u}_{x2})(\mathfrak{u}_x,\mathfrak{v}_{y2}) = \dfrac{\xi_{E1}(\mathfrak{u}_x,\mathfrak{u}_x)\xi_{E2}(\mathfrak{u}_{x2},\mathfrak{v}_{y2})}{1+\Big(\big(1-\xi_{E1}(\mathfrak{u}_x,\mathfrak{u}_x)\big)\big(1-\xi_{E2}(\mathfrak{u}_{x2},\mathfrak{v}_{y2})\big)\Big)},$
for all $\mathfrak{u}_x\in\mathfrak{V}_{V1}$, *for all* $(\mathfrak{u}_{x2},\mathfrak{v}_{y2})\in\mathfrak{E}_{E2}$ *and where,*

$$\xi_{E1}(\mathfrak{u}_x,\mathfrak{u}_x) = \mathfrak{T}_T\Big(\big[\psi_{V1}(\mathfrak{u}_x)\big]^n, \big[\psi_{V1}(\mathfrak{u}_x)\big]^n\Big),$$

with n *representing the number of edges meeting at* $\mathfrak{u}_x$ *in* $\mathfrak{G}_{G1}$.

iii) $(\xi_{E1}\Box_M\xi_{E2})(\mathfrak{u}_{x1},\mathfrak{w}_y)(\mathfrak{v}_{y1},\mathfrak{w}_y) = \dfrac{\xi_{E1}(\mathfrak{u}_{x1},\mathfrak{v}_{y1})\xi_{E2}(\mathfrak{w}_y,\mathfrak{w}_y)}{1+\Big(\big(1-\xi_{E1}(\mathfrak{u}_{x1},\mathfrak{v}_{y1})\big)\big(1-\xi_{E2}(\mathfrak{w}_y,\mathfrak{w}_y)\big)\Big)},$
for all $(\mathfrak{u}_{x1},\mathfrak{v}_{y1})\in\mathfrak{E}_{E1}$ *for all* $\mathfrak{w}_y\in\mathfrak{V}_{V2}$ *and where,*

$$\xi_{E2}(\mathfrak{w}_y,\mathfrak{w}_y) = \mathfrak{T}_T\Big(\big[\psi_{V1}(\mathfrak{w}_y)\big]^n, \big[\psi_1(\mathfrak{w}_y)\big]^n\Big),$$

with n *representing the number of edges meeting at* $\mathfrak{w}_y$ *in* $\mathfrak{G}_{G2}$.

Remark 3.2. *The product of two Einstein fuzzy graphs that satisfy conditions* (i) *and* (ii) *of the M-Cartesian product is called the* ***Quasi*** I_M ***Cartesian product****, denoted by* $\mathcal{G}_{G_1} \sqsubset_{Q_M} \mathcal{G}_{G_2}$. *Similarly, the product of two Einstein fuzzy graphs that satisfy conditions* (i) *and* (iii) *of the M-Cartesian product is called the* ***Quasi*** II_M ***Cartesian product****, denoted by* $\mathcal{G}_{G_1} \sqsupset_Q \mathcal{G}_{G_2}$.

Example 3.2. *Let* $\mathcal{G}_{G1} = (\psi_{V1},\xi_{E1})$ *and* $\mathcal{G}_{G2} = (\psi_{V2},\xi_{E2})$ *be two Einstein fuzzy graphs with the following membership values:*

$$\psi_{V1} = \left\{\frac{\mathfrak{a}_{u_1}}{0.8}, \frac{\mathfrak{b}_{u_2}}{0.6}, \frac{\mathfrak{c}_{u_3}}{0.9}\right\},$$
$$\xi_{E1} = \left\{\frac{(\mathfrak{a}_{u_1},\mathfrak{b}_{u_2})}{0.4}, \frac{(\mathfrak{b}_{u_2},\mathfrak{c}_{u_3})}{0.5}\right\},$$
$$\psi_{V2} = \left\{\frac{\mathfrak{l}_{v_1}}{0.7}, \frac{\mathfrak{m}_{v_2}}{0.5}, \frac{\mathfrak{n}_{v_3}}{0.6}, \frac{o_{v_4}}{0.4}\right\},$$
$$\xi_{E2} = \left\{\frac{(\mathfrak{l}_{v_1},\mathfrak{m}_{v_2})}{0.3}, \frac{(\mathfrak{m}_{v_2},\mathfrak{n}_{v_3})}{0.2}, \frac{(\mathfrak{n}_{v_3},\mathfrak{o}_{v_4})}{0.1}\right\},$$

the M-Cartesian product $\mathcal{G}_{G1}\Box_M\mathcal{G}_{G2}$ *is represented in Figures 5, 6, 7 and 8.*

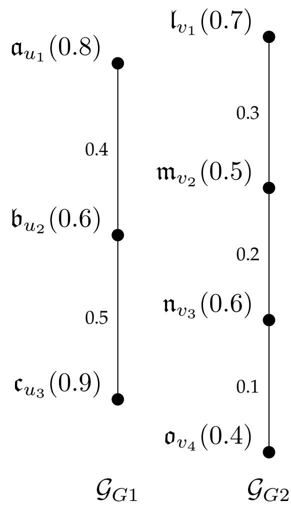

Figure 5: Einstein fuzzy graphs.

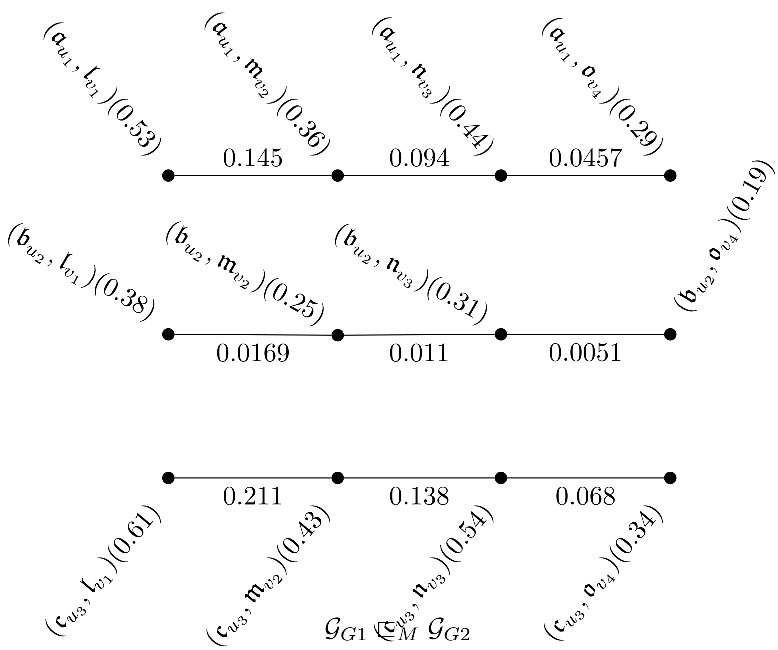

Figure 6: Einstein fuzzy graph: $\mathcal{G}_{G1} \sqsubset_M \mathcal{G}_{G2}$.

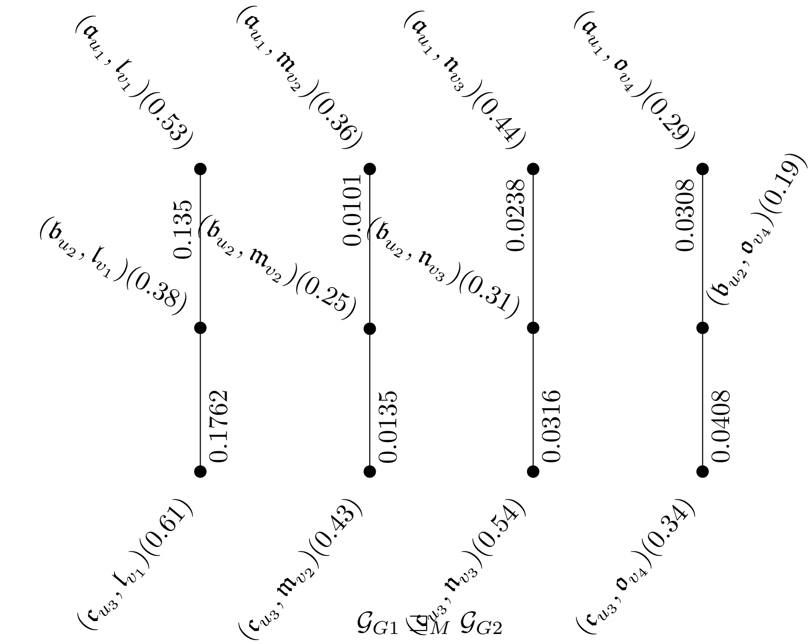

Figure 7: Einstein fuzzy graph: $\mathcal{G}_{G1} \sqsupset_M \mathcal{G}_{G2}$.

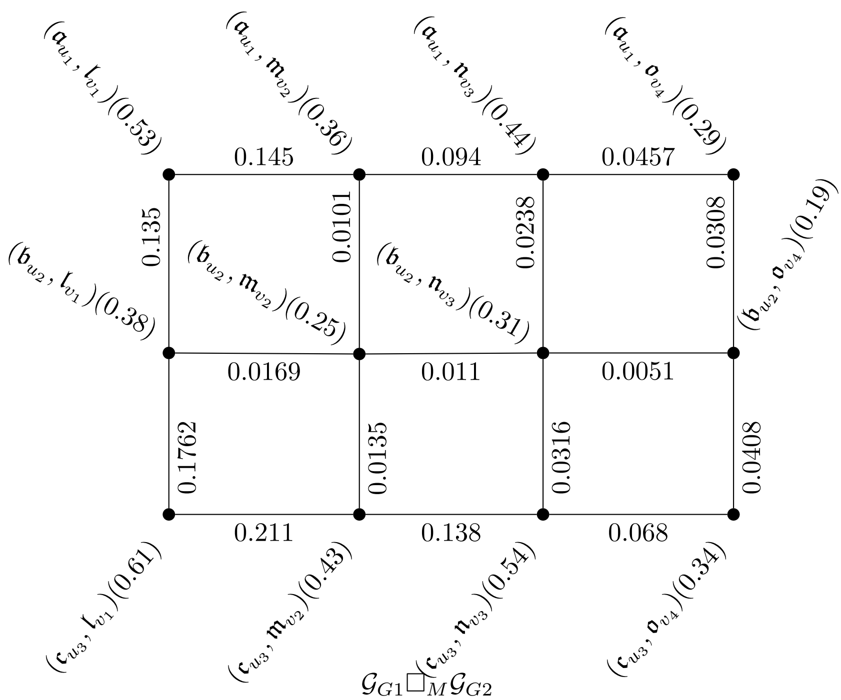

Figure 8: Einstein fuzzy graph: $\mathcal{G}_{G1} \square_M \mathcal{G}_{G2}$.

Theorem 3.1. *The "M-Cartesian product" $\mathcal{G}_{G1}\square_M\mathcal{G}_{G2}$ of $\mathcal{G}_{G1}$ and $\mathcal{G}_{G2}$ is the Einstein fuzzy graph associated with the Cartesian product $\mathfrak{G}_{G1}\square\mathfrak{G}_{G2}$, where $\mathcal{G}_{G1}$ and $\mathcal{G}_{G2}$ represent the Einstein fuzzy graphs associated with the graphs $\mathfrak{G}_{G1}$ and $\mathfrak{G}_{G2}$, respectively.*

Proof. Take any $\mathfrak{u}_x \in \mathfrak{V}_{V1}$, $(\mathfrak{u}_2, \mathfrak{v}_2) \in \mathfrak{E}_2$. Then,

$$(\xi_{E1}\square\xi_{E2})\Big((\mathfrak{u}_x,\mathfrak{u}_{x2})(\mathfrak{u}_x,\mathfrak{v}_{y2})\Big) = \mathfrak{T}_T\Big(\xi_{E1}(\mathfrak{u}_x,\mathfrak{u}_x),\xi_{E2}(\mathfrak{u}_{x2},\mathfrak{v}_{y2})\Big),$$

where $\mathfrak{T}_T(\mathfrak{u}_x,\mathfrak{v}_y) = \dfrac{\mathfrak{u}_x\mathfrak{v}_y}{1+(1-\mathfrak{u}_x)(1-\mathfrak{v}_y)}$. Thus,

$$\begin{aligned}
&\le \mathfrak{T}_T\Big(\mathfrak{T}_T\big([\psi_1(\mathfrak{u}_x)]^n,[\psi_{V1}(\mathfrak{u}_x)]^n\big),\mathfrak{T}_T\big(\psi_{V2}(\mathfrak{u}_{x2}),\psi_{V2}(\mathfrak{v}_{y2})\big)\Big)\\
&\le \mathfrak{T}_T\Big(\mathfrak{T}_T\Big(\psi_{V1}(\mathfrak{u}_x),(\psi_{V1}(\mathfrak{u}_x)),\mathfrak{T}_T\big(\psi_{V2}(\mathfrak{u}_{x2}),\psi_2(\mathfrak{v}_{y2})\big)\Big)\Big)\\
&\qquad\Big(\text{since } \xi_{E1}(\mathfrak{u}_x,\mathfrak{u}_x)=\mathfrak{T}_T\big([\psi_{V1}(\mathfrak{u}_x)]^n,[\psi_{V1}(\mathfrak{u}_x)]^n\big)\le\mathfrak{T}_T\big(\psi_{V1}(\mathfrak{u}_x),\psi_{V1}(\mathfrak{u}_x)\big)\Big)\\
&\le \mathfrak{T}_T\left(\frac{\psi_{V1}(\mathfrak{u}_x)\psi_{V1}(\mathfrak{u}_x)}{1+(1-\psi_{V1}(\mathfrak{u}_x))(1-\psi_{V1}(\mathfrak{u}_x))},\frac{\psi_{V2}(\mathfrak{u}_{\mathfrak{x}2})\psi_{V2}(\mathfrak{v}_{y2})}{1+(1-\psi_{V2}(\mathfrak{u}_{x2}))(1-\psi_{V2}(\mathfrak{v}_{y2}))}\right)
\end{aligned}$$

put $\psi_{V1}(\mathfrak{u}_x)=l$, $\psi_{V2}(\mathfrak{u}_{x2})=o$, $\psi_{V2}(\mathfrak{v}_{y2})=p$.

Hence,

$$\begin{aligned}
&(\xi_{E1}\square_M\xi_{E2})\Big((\mathfrak{u}_x,\mathfrak{u}_{x2})(\mathfrak{u}_x,\mathfrak{v}_{y2})\Big)\\
&\quad\le \mathfrak{T}_T\left(\frac{l^2}{1+(1-l)(1-l)},\frac{op}{1+(1-o)(1-p)}\right)\\
&\quad\le\left[\frac{\dfrac{(l^2)(op)}{[1+(1-l)(1-l)][1+(1-p)(1-o)]}}{1+\left[\left(1-\dfrac{l^2}{1+(1-l)(1-l)}\right)\left(1-\dfrac{op}{1+(1-o)(1-p)}\right)\right]}\right]\\
&\quad\le\left[\frac{\dfrac{(lo)(lp)}{[1+(1-l)(1-o)][1+(1-l)(1-p)]}}{1+\left[\left(1-\dfrac{l0}{1+(1-l)(1-o)}\right)\left(1-\dfrac{lp}{1+(1-l)(1-p)}\right)\right]}\right]\\
&\quad\le\left[\frac{\left[\dfrac{lo}{1+(1-l)(1-o)}\right]\left[\dfrac{lp}{1+(1-l)(1-p)}\right]}{1+\left[\left(1-\dfrac{l0}{1+(1-l)(1-o)}\right)\left(1-\dfrac{lp}{1+(1-l)(1-p)}\right)\right]}\right]\\
&\quad\le\left[\frac{\left[\dfrac{\psi_{V1}(\mathfrak{u}_x)\psi_2(\mathfrak{u}_{x2})}{1+(1-\psi_{V1}(\mathfrak{u}_x))(1-\psi_{V2}(\mathfrak{u}_{x2}))}\right]\left[\dfrac{\psi_{V1}(\mathfrak{u}_x)\psi_2(\mathfrak{v}_{y2})}{1+(1-\psi_{V1}(\mathfrak{u}_x))(1-\psi_{V2}(\mathfrak{v}y2))}\right]}{1+\left[\left(1-\dfrac{\psi_{V1}(\mathfrak{u}_x)\psi_{V2}(\mathfrak{u}_{x2})}{1+(1-\psi_{V1}(\mathfrak{u}_x))(1-\psi_{V2}(\mathfrak{u}_{x2}))}\right)\left(1-\dfrac{\psi_{V1}(\mathfrak{u}_x)\psi_{V2}(\mathfrak{v}_{y2})}{1+(1-\psi_{V1}(\mathfrak{u}_x))(1-\psi_{V2}(\mathfrak{v}_{y2}))}\right)\right]}\right]\\
&\quad\le\frac{(\psi_{V1}\square_M\psi_{V2})(\mathfrak{u}_x,\mathfrak{u}_{x2})(\psi_{V1}\square_M\psi_{V2})(\mathfrak{u}_x,\mathfrak{v}_{y2})}{1+\Big[1-(\psi_{V1}\square_M\psi_{V2})(\mathfrak{u}_x,\mathfrak{u}_{x2})\Big]\Big[1-(\psi_{V1}\square_M\psi_{V2})(\mathfrak{u}_x,\mathfrak{v}_{y2})\Big]},
\end{aligned}$$

i.e.,

$$(\xi_{E1}\square_M\xi_{E2})\Big((\mathfrak{u}_x,\mathfrak{u}_{x2})(\mathfrak{u}_x,\mathfrak{v}_{y2})\Big) \leq \frac{(\psi_{V1}\square_M\psi_{V2})(\mathfrak{u}_x,\mathfrak{u}_{x2})(\psi_{V1}\square_M\psi_{V2})(\mathfrak{u}_x,\mathfrak{v}_{y2})}{1+\Big[1-(\psi_{V1}\square_M\psi_{V2})(\mathfrak{u}_x,\mathfrak{u}_{x2})\Big]\Big[1-(\psi_{V1}\square_M\psi_{V2})(\mathfrak{u}_x,\mathfrak{v}_{y2})\Big]}.$$

Again, take any $\mathfrak{w}_y \in V_2$, $(\mathfrak{u}_{x1},\mathfrak{v}_{y1}) \in \mathfrak{E}_{E1}$. Then,

$$(\xi_{E1}\square_M\xi_{E2})\Big((\mathfrak{u}_{x1},\mathfrak{w}_y)(\mathfrak{v}_{y1},\mathfrak{w}_y)\Big) = \mathfrak{T}_T\Big(\xi_{E1}(\mathfrak{u}_{x1},\mathfrak{v}_{y1}),\xi_{E2}(\mathfrak{w}_y,\mathfrak{w}_y)\Big),$$

where $\mathfrak{T}_T(\mathfrak{u}_x,\mathfrak{v}_y) = \dfrac{\mathfrak{u}_x\mathfrak{v}_y}{1+(1-\mathfrak{u}_x)(1-\mathfrak{v}_y)}$

$$\begin{aligned}
&\leq \mathfrak{T}_T\Big(\mathfrak{T}_T\big(\psi_{V1}(\mathfrak{u}_{x1}),\psi_{V1}(\mathfrak{v}_{y1})\big),\mathfrak{T}_T([\psi_{V2}(\mathfrak{w}_y)]^n),[\psi_{V2}(\mathfrak{w}_y)]^n\Big)\\
&\leq \mathfrak{T}_T\Big(\mathfrak{T}_T\big(\psi_{V1}(\mathfrak{u}_{x1}),\psi_{V1}(\mathfrak{v}_{y1})\big),\mathfrak{T}_T\big(\psi_{V2}(\mathfrak{w}_y),\psi_{V2}(\mathfrak{w}_y)\big)\Big)\\
&\quad\Big(\text{since } \xi_{V2}(\mathfrak{w}_y,\mathfrak{w}_y) = \mathfrak{T}_T\Big([\psi_{V2}(\mathfrak{w}_y)]^n,[\psi_{V2}(\mathfrak{w}_y)]^n\Big) \leq \mathfrak{T}_T\Big(\psi_{V2}(\mathfrak{w}_y),\psi_{V2}(\mathfrak{w}_y)\Big)\Big)\\
&\leq \mathfrak{T}_T\left(\frac{\psi_{V1}(\mathfrak{u}_{x1})\psi_{V1}(\mathfrak{v}_{y1})}{1+(1-\psi_{V1}(\mathfrak{u}_{x1}))(1-\psi_{V1}(\mathfrak{v}_{y1}))},\frac{\psi_{V2}(\mathfrak{w}_y)\psi_{V2}(\mathfrak{w}_y)}{1+(1-\psi_{V2}(\mathfrak{w}_y))(1-\psi_{V2}(\mathfrak{w}_y))}\right)
\end{aligned}$$

put $\psi_{V1}(\mathfrak{u}_{x1}) = l$, $\psi_{V1}(\mathfrak{v}_{y1}) = m$, $\psi_{V2}(\mathfrak{w}_{y]}) = p$.

Hence,

$$\begin{aligned}
&(\xi_{E1}\square_M\xi_{E2})\Big((\mathfrak{u}_{x1},\mathfrak{w}_y)(\mathfrak{v}_{x1},\mathfrak{w}_y)\Big)\\
&\leq \mathfrak{T}_T\left(\frac{lm}{1+(1-l)(1-m)},\frac{p^2}{1+(1-p)(1-p)}\right)\\
&\leq \left[\frac{\dfrac{(lm)(p^2)}{[1+(1-l)(1-m)][1+(1-p)(1-p)]}}{1+\left[\left(1-\dfrac{lm}{1+(1-l)(1-m)}\right)\left(1-\dfrac{p^2}{1+(1-p)(1-p)}\right)\right]}\right]\\
&\leq \left[\frac{\dfrac{(lp)(mp)}{[1+(1-l)(1-p)][1+(1-m)(1-p)]}}{1+\left[\left(1-\dfrac{lp}{1+(1-l)(1-p)}\right)\left(1-\dfrac{mp}{1+(1-m)(1-p)}\right)\right]}\right]\\
&\leq \left[\frac{\left[\dfrac{lp}{1+(1-l)(1-p)}\right]\left[\dfrac{mp}{1+(1-m)(1-p)}\right]}{1+\left[\left(1-\dfrac{lp}{1+(1-l)(1-p)}\right)\left(1-\dfrac{mp}{1+(1-m)(1-p)}\right)\right]}\right]\\
&\leq \left[\frac{\left[\dfrac{\psi_{V1}(\mathfrak{u}_{x1})\psi_{V2}(\mathfrak{w}_y)}{1+(1-\psi_{V1}(\mathfrak{u}_{x1}))(1-\psi_{V2}(\mathfrak{w}_y))}\right]\left[\dfrac{\psi_{V1}(\mathfrak{v}_{y1})\psi_{V2}(\mathfrak{w}_y)}{1+(1-\psi_{V1}(\mathfrak{v}_{y1}))(1-\psi_{V2}(\mathfrak{w}_y))}\right]}{1+\left[\left(1-\dfrac{\psi_{V1}(\mathfrak{u}_{x1})\psi_{V2}(\mathfrak{w}_y)}{1+(1-\psi_{V1}(\mathfrak{u}_{x1}))(1-\psi_{V2}(\mathfrak{w}_y))}\right)\left(1-\dfrac{\psi_{V1}(\mathfrak{v}_{y1})\psi_{V2}(\mathfrak{w}_y)}{1+(1-\psi_{V1}(\mathfrak{v}_{y1}))(1-\psi_{V2}(\mathfrak{w}_y))}\right)\right]}\right]\\
&\leq \frac{(\psi_{V1}\square_M\psi_{V2})(\mathfrak{u}_{x1},\mathfrak{w}_y)(\psi_{V1}\square_M\psi_{V2})(\mathfrak{v}_{y1},\mathfrak{w}_y)}{1+\Big[1-(\psi_{V1}\square_M\psi_{V2})(\mathfrak{u}_{x1},\mathfrak{w}_y)\Big]\Big[1-(\psi_{V1}\square_M\psi_{V2})(\mathfrak{v}_{y1},\mathfrak{w}_y)\Big]},
\end{aligned}$$

i.e.,

$$(\xi_{E1}\Box_M\xi_{E2})\big((\mathfrak{u}_{x1},\mathfrak{w}_y)(\mathfrak{v}_{y1},\mathfrak{w}_y)\big) \leq \frac{(\psi_{V1}\Box_M\psi_{V2})(\mathfrak{u}_{x1},\mathfrak{w}_y)(\psi_{V1}\Box_M\psi_{V2})(\mathfrak{v}_{y1},\mathfrak{w}_y)}{1+\Big[1-(\psi_{V1}\Box_M\psi_{V2})(\mathfrak{u}_{x1},\mathfrak{w}_y)\Big]\Big[1-(\psi_{V1}\Box_M\psi_{V2})(\mathfrak{v}_{y1},\mathfrak{w}_y)\Big]}.$$

This completes the proof. □

Corollary 3.1. *The **Quasi** I_M **Cartesian product** $\mathcal{G}_{G_1} \sqsubset_{Q_M} \mathcal{G}_{G_2}$ and the **Quasi** II_M **Cartesian product** $\mathcal{G}_{G_1} \sqsupset_{Q_M} \mathcal{G}_{G_2}$ of the Einstein fuzzy graphs $\mathcal{G}_{G_1}$ and $\mathcal{G}_{G_2}$ are Einstein fuzzy graphs.*

Proof. The result follows immediately from the above theorem. □

Proposition 3.1 (Properties of the M-Cartesian Product)**.** *The M-Cartesian product $\mathcal{G}_{G_1}\Box_M\mathcal{G}_{G_2}$ possesses the following properties:*

- *It includes all the edges of the Quasi I_M Cartesian product $\mathcal{G}_{G_1} \sqsubset_M \mathcal{G}_{G_2}$, preserving the corresponding membership values.*
- *It includes all the edges of the Quasi II_M Cartesian product $\mathcal{G}_{G_1} \sqsupset_M \mathcal{G}_{G_2}$, also with the same membership values.*
- *It can be regarded as the union of the edge sets of the Quasi I_M and Quasi II_M Cartesian products and includes the same vertex set with the same membership values.*

Definition 3.4. *The **Modified semi-strong product** or **M-semi-strong product** of two Einstein fuzzy graphs,*

$$\mathcal{G}_{G_1} = (\psi_{V_1}, \xi_{E_1}) \text{ of } \mathfrak{G}_{G_1} = (\mathfrak{V}_{V_1}, \mathfrak{E}_{E_1}) \quad \text{and} \quad \mathcal{G}_{G_2} = (\psi_{V_2}, \xi_{E_2}) \text{ of } \mathfrak{G}_{G_2} = (\mathfrak{V}_{V_2}, \mathfrak{E}_{E_2}),$$

denoted by,

$$\mathcal{G}_{G_1} \bullet_M \mathcal{G}_{G_2} = (\psi_{V_1} \bullet_M \psi_{V_2}, \xi_{E_1} \bullet_M \xi_{E_2}),$$

is defined as follows. Here, ψ_{V_i} and ξ_{E_i} are fuzzy subsets of $\mathfrak{V}_{V_i}$ and $\mathfrak{E}_{E_i}$, respectively, for $i \in \{1, 2\}$:

i) $(\psi_{V_1} \bullet_M \psi_{V_2})(\mathfrak{u}_{x_1}, \mathfrak{u}_{x_2}) = \dfrac{\psi_{V_1}(\mathfrak{u}_{x_1})\,\psi_{V_2}(\mathfrak{u}_{x_2})}{1+\Big(1-\psi_{V_1}(\mathfrak{u}_{x_1})\Big)\Big(1-\psi_{V_2}(\mathfrak{u}_{x_2})\Big)},$
for all $(\mathfrak{u}_{x_1}, \mathfrak{u}_{x_2}) \in \mathfrak{V}_{V_1} \times \mathfrak{V}_{V_2}$.

ii) $\big((\mathfrak{u}_x, \mathfrak{u}_{x_2}), (\mathfrak{u}_x, \mathfrak{v}_{y_2})\big)$:

$$(\xi_{E_1} \bullet_M \xi_{E_2})((\mathfrak{u}_x, \mathfrak{u}_{x_2}), (\mathfrak{u}_x, \mathfrak{v}_{y_2})) = \frac{\xi_{E_1}(\mathfrak{u}_x, \mathfrak{u}_x)\,\xi_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2})}{1+\Big(1-\xi_{E_1}(\mathfrak{u}_x, \mathfrak{u}_x)\Big)\Big(1-\xi_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2})\Big)},$$

for all $\mathfrak{u}_x \in \mathfrak{V}_{V_1}$ *and* $(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}) \in \mathfrak{E}_{E_2}$ *where,*

$$\xi_{E_1}(\mathfrak{u}_x, \mathfrak{u}_x) = \mathfrak{T}_T\left([\psi_{V_1}(\mathfrak{u}_x)]^n, [\psi_{E_1}(\mathfrak{u}_x)]^n\right),$$

and n is the number of edges incident to $\mathfrak{u}_x$ in $\mathfrak{G}_{G_1}$.

iii) $(\xi_{E_1} \bullet_M \xi_{E_2})\Big((\mathfrak{u}_{x_1}, \mathfrak{u}_{x_2}), (\mathfrak{v}_{y_1}, \mathfrak{v}_{y_2})\Big) = \dfrac{\xi_{E_1}(\mathfrak{u}_{x_1}, \mathfrak{v}_{y_1})\,\xi_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2})}{1+\Big(1-\xi_{E_1}(\mathfrak{u}_{x_1}, \mathfrak{v}_{y_1})\Big)\Big(1-\xi_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2})\Big)},$
for all $(\mathfrak{u}_{x_1}, \mathfrak{v}_{y_1}) \in \mathfrak{E}_{E_1}$ *and* $(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}) \in \mathfrak{E}_{E_2}$.

Example 3.3. *Consider the Einstein fuzzy graphs $\mathcal{G}_{G1}$ and $\mathcal{G}_{G2}$ as given in Example 3.1, where the M-semi-strong product $\mathcal{G}_{G1} \bullet_M \mathcal{G}_{G2}$ is depicted in Figure 9.*

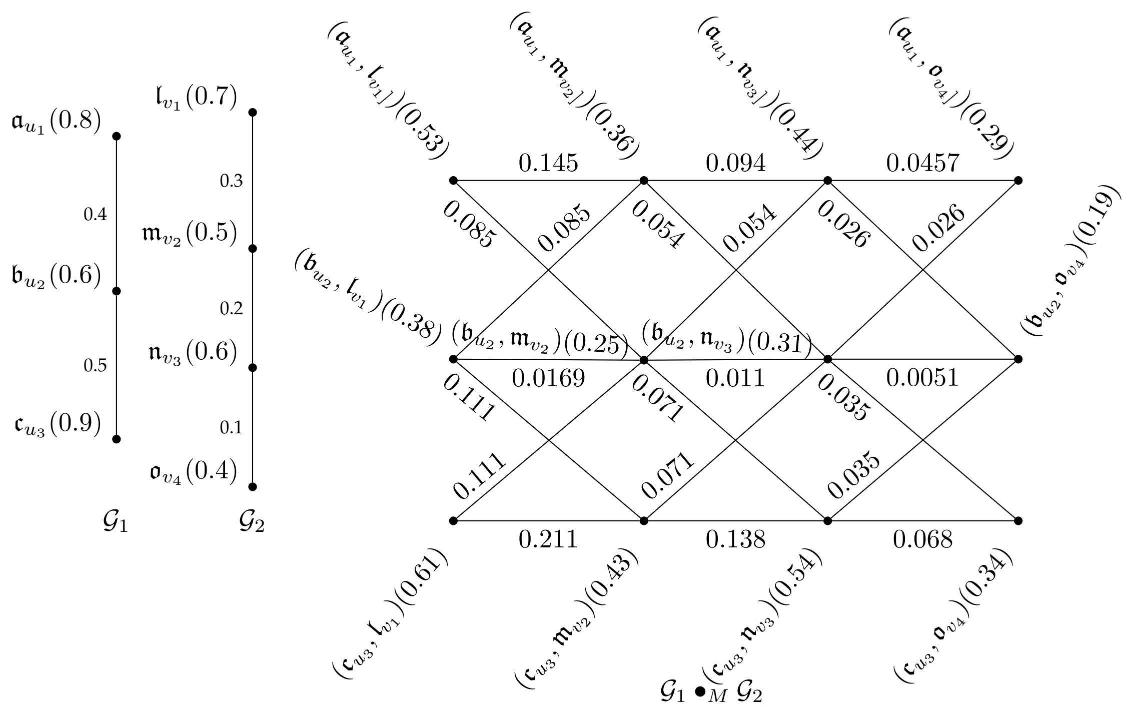

Figure 9: Einstein fuzzy graphs: $\mathcal{G}_1$, $\mathcal{G}_2$ $\mathcal{G}_1 \bullet_M \mathcal{G}_2$.

Theorem 3.2. *The* ***M-semi-strong product*** *$\mathcal{G}_{G_1} \bullet_M \mathcal{G}_{G_2}$ of $\mathcal{G}_{G_1}$ and $\mathcal{G}_{G_2}$ is the Einstein fuzzy graph of $\mathfrak{G}_{G_1} \bullet \mathfrak{G}_{G_2}$. Here, $\mathcal{G}_{G_1}$ and $\mathcal{G}_{G_2}$ are the Einstein fuzzy graphs corresponding to the graphs $\mathfrak{G}_{G_1}$ and $\mathfrak{G}_{G_2}$, respectively.*

Proof. The proof of this theorem can be found in Theorem 3.1 and [11]. □

Definition 3.5. *The* ***M-strong product*** *$\mathcal{G}_{G1} \boxtimes_M \mathcal{G}_{G2} = (\psi_{V1} \boxtimes_M \psi_{V2}, \xi_{E1} \boxtimes_M \xi_{E2})$ of the Einstein fuzzy graphs,*

$$\mathcal{G}_{G1} = (\psi_{V1}, \xi_{E1}) \text{ of } \mathfrak{G}_{G1} = (\mathfrak{V}_{V1}, \mathfrak{E}_{E1}) \quad \text{and} \quad \mathcal{G}_{G2} = (\psi_{V2}, \xi_{E2}) \text{ of } \mathfrak{G}_{G2} = (\mathfrak{V}_{V2}, \mathfrak{E}_{E2}),$$

is defined as follows. Here, ψ_{V_i} and ξ_{E_i} are fuzzy subsets of $\mathfrak{V}_{V_i}$ and $\mathfrak{E}_{E_i}$, respectively, for $i \in \{1, 2\}$;

i) $$(\psi_{V1} \boxtimes_M \psi_{V2})(\mathfrak{u}_{x1}, \mathfrak{u}_{x2}) = \frac{\psi_{V1}(\mathfrak{u}_{x1})\psi_{V2}(\mathfrak{u}_{x2})}{1 + \Big(1 - \psi_{V1}(\mathfrak{u}_{x1})\Big)\Big(1 - \psi_{V2}(\mathfrak{u}_{x2})\Big)},$$
for all $(\mathfrak{u}_{x1}, \mathfrak{u}_{x2}) \in \mathfrak{V}_{V1} \times \mathfrak{V}_{V2}$.

ii) $$(\xi_{E1} \boxtimes_M \xi_{E2})\Big((\mathfrak{u}_x, \mathfrak{u}_{x2}), (\mathfrak{u}_x, \mathfrak{v}_{y2})\Big) = \frac{\xi_{E1}(\mathfrak{u}_x, \mathfrak{u}_x)\xi_{E2}(\mathfrak{u}_{x2}, \mathfrak{v}_{y2})}{1 + \Big(1 - \xi_{E1}(\mathfrak{u}_x, \mathfrak{u}_x)\Big)\Big(1 - \xi_{E2}(\mathfrak{u}_{x2}, \mathfrak{v}_{y2})\Big)},$$
for all $\mathfrak{u}_x \in \mathfrak{V}_{V1}$, $(\mathfrak{u}_{x2}, \mathfrak{v}_{y2}) \in \mathfrak{E}_{E2}$ and where,

$$\xi_{E1}(\mathfrak{u}_x, \mathfrak{u}_x) = \mathfrak{T}_T\Big(\big[\psi_{V1}(\mathfrak{u}_x)\big]^n, \big[\psi_{V1}(\mathfrak{u}_x)\big]^n\Big),$$

with n representing the number of edges incident to $\mathfrak{u}_x$ in $\mathfrak{G}_{G1}$.

iii) $(\xi_{E1} \boxtimes_M \xi_{E2})\Big((\mathfrak{u}_{x1}, \mathfrak{w}_y), (\mathfrak{v}_{y1}, \mathfrak{w}_y)\Big) = \dfrac{\xi_{E1}(\mathfrak{u}_{x1}, \mathfrak{v}_{y1})\xi_{E2}(\mathfrak{w}_y, \mathfrak{w}_y)}{1 + \Big(1 - \xi_{E1}(\mathfrak{u}_{x1}, \mathfrak{v}_{y1})\Big)\Big(1 - \xi_{E2}(\mathfrak{w}_y, \mathfrak{w}_y)\Big)},$

for all $(\mathfrak{u}_{x1}, \mathfrak{v}_{y1}) \in \mathfrak{E}_{E1}$, $\mathfrak{w}_y \in \mathfrak{V}_{V2}$ *and where,*

$$\xi_{E2}(\mathfrak{w}_y, \mathfrak{w}_y) = \mathfrak{T}_T\Big(\big[\psi_{V2}(\mathfrak{w}_y)\big]^n, \big[\psi_{V2}(\mathfrak{w}_y)\big]^n\Big),$$

with n representing the number of edges incident to $\mathfrak{w}_y$ in $\mathfrak{G}_{G2}$.

iv) $(\xi_{E1} \boxtimes_M \xi_{E2})\Big((\mathfrak{u}_{x1}, \mathfrak{u}_{x2}), (\mathfrak{v}_{y1}, \mathfrak{v}_{y2})\Big) = \dfrac{\xi_{E1}(\mathfrak{u}_{x1}, \mathfrak{v}_{y1})\xi_{E2}(\mathfrak{u}_{x2}, \mathfrak{v}_{y2})}{1 + \Big(1 - \xi_{E1}(\mathfrak{u}_{x1}, \mathfrak{v}_{y1})\Big)\Big(1 - \xi_{E2}(\mathfrak{u}_{x2}, \mathfrak{v}_{y2})\Big)},$

for all $(\mathfrak{u}_{x1}, \mathfrak{v}_{y1}) \in \mathfrak{E}_{E1}$, $(\mathfrak{u}_{x2}, \mathfrak{v}_{y2}) \in \mathfrak{E}_{E2}$.

Example 3.4. *The "Einstein fuzzy graphs" $\mathcal{G}_{G1}$ and $\mathcal{G}_{G2}$ as described in "Example 3.1". The M-strong product $\mathcal{G}_{G1} \boxtimes_M \mathcal{G}_{G2}$ is depicted in Figure 10.*

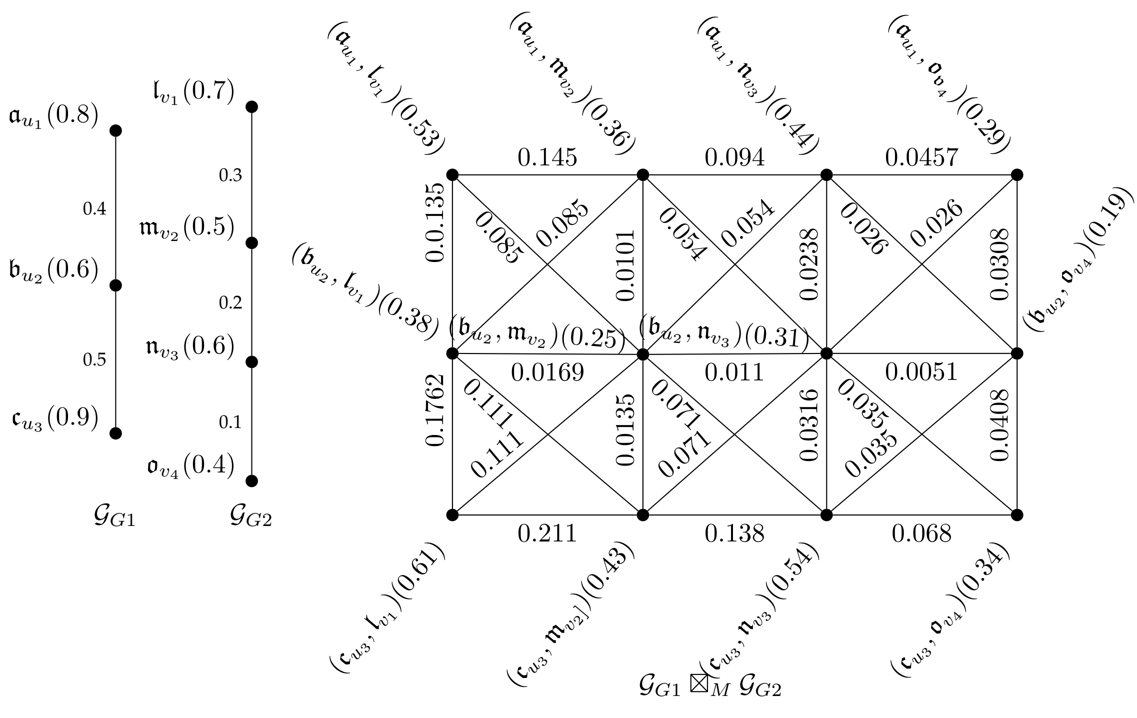

Figure 10: Einstein fuzzy graphs: $\mathcal{G}_{G1}, \mathcal{G}_{G2}, \mathcal{G}_{G1} \boxtimes_M \mathcal{G}_{G2}$.

Theorem 3.3. *The M-strong product $\mathcal{G}_{G1} \boxtimes_M \mathcal{G}_{G2}$ of $\mathcal{G}_{G1}$ and $\mathcal{G}_{G2}$ is the Einstein fuzzy graph of $\mathfrak{G}_{G1} \boxtimes \mathfrak{G}_{G2}$. Here, $\mathcal{G}_{G1}$ and $\mathcal{G}_{G2}$ are the Einstein fuzzy graphs of the graphs $\mathfrak{G}_{G1}$ and $\mathfrak{G}_{G2}$, respectively.*

Proof. This theorem can be easily found out from the above theorems and [11]. □

Definition 3.6. *The **Modified lexicographic product** or **M-lexicographic product** of two Einstein fuzzy graphs,*

$$\mathcal{G}_{G_1} = (\psi_{V_1}, \xi_{E_1}) \;\; of \;\; \mathfrak{G}_{G_1} = (\mathfrak{V}_{V_1}, \mathfrak{E}_{E_1}), \;\; and \;\; \mathcal{G}_{G_2} = (\psi_{V_2}, \xi_{E_2}) \;\; of \;\; \mathfrak{G}_{G_2} = (\mathfrak{V}_{V_2}, \mathfrak{E}_{E_2}),$$

denoted by,

$$\mathcal{G}_{G_1}[\mathcal{G}_{G_2}]_M = (\psi_{V_1} \circ_M \psi_{V_2}, \xi_{E_1} \circ_M \xi_{E_2}),$$

is defined as follows. Here, ψ_{V_i} *and* ξ_{E_i} *are fuzzy subsets of* $\mathfrak{V}_{V_i}$ *and* $\mathfrak{E}_{E_i}$*, respectively, for* $i \in \{1, 2\}$*:*

i) $(\psi_{V_1} \circ_M \psi_{V_2})(\mathfrak{u}_{x_1}, \mathfrak{u}_{x_2}) = \dfrac{\psi_{V_1}(\mathfrak{u}_{x_1})\,\psi_{V_2}(\mathfrak{u}_{x_2})}{1 + \Big(1 - \psi_{V_1}(\mathfrak{u}_{x_1})\Big)\Big(1 - \psi_{V_2}(\mathfrak{u}_{x_2})\Big)},$
for all $(\mathfrak{u}_{x_1}, \mathfrak{u}_{x_2}) \in \mathfrak{V}_{V_1} \times \mathfrak{V}_{V_2}$.

ii) For edges of the form $\Big((\mathfrak{u}_x, \mathfrak{u}_{x_2}), (\mathfrak{u}_x, \mathfrak{v}_{y_2})\Big)$,

$$(\xi_{E_1} \circ_M \xi_{E_2})\Big((\mathfrak{u}_x, \mathfrak{u}_{x_2}), (\mathfrak{u}_x, \mathfrak{v}_{y_2})\Big) = \frac{\xi_{E_1}(\mathfrak{u}_x, \mathfrak{u}_x)\,\xi_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2})}{1 + \Big(1 - \xi_{E_1}(\mathfrak{u}_x, \mathfrak{u}_x)\Big)\Big(1 - \xi_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2})\Big)},$$

for all $\mathfrak{u}_x \in \mathfrak{V}_{V_1}$ *and* $(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}) \in \mathfrak{E}_{E_2}$ *where,*

$$\xi_{E_1}(\mathfrak{u}_x, \mathfrak{u}_x) = \mathfrak{T}_T\Big(\big[\psi_{V_1}(\mathfrak{u}_x)\big]^{n_{u_x}}, \big[\psi_{V_1}(\mathfrak{u}_x)\big]^{n_{u_x}}\Big),$$

and n_{u_x} *denotes the number of edges incident on the vertex* $\mathfrak{u}_x$ *in* $\mathfrak{G}_{G_1}$*.*

iii) For edges of the form $((\mathfrak{u}_{x_1}, \mathfrak{u}_{x_2}), (\mathfrak{v}_{y_1}, \mathfrak{v}_{y_2}))$,

$$(\xi_{E_1} \circ_M \xi_{E_2})((\mathfrak{u}_{x_1}, \mathfrak{u}_{x_2}), (\mathfrak{v}_{y_1}, \mathfrak{v}_{y_2})) = \frac{\xi_{E_1}(\mathfrak{u}_{x_1}, \mathfrak{v}_{y_1})\,\xi^M_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2})}{1 + (1 - \xi_{E_1}\Big(\mathfrak{u}_{x_1}, \mathfrak{v}_{y_1})\Big)\Big(1 - \xi^M_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2})\Big)},$$

for all $(\mathfrak{u}_{x_1}, \mathfrak{v}_{y_1}) \in \mathfrak{E}_{E_1}$ *and* $(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}) \in \mathfrak{V}_{V_2} \times \mathfrak{V}_{V_2}$*, where,*

$$\xi^M_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}) = \begin{cases} \xi_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}), & \text{if } (\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}) \in \mathfrak{E}_{E_2}, \\ \mathfrak{T}_T\Big([\psi_{V_2}(\mathfrak{u}_{x_2})]^{n_{u_{x_2}}}, [\psi_{V_2}(\mathfrak{v}_{y_2})]^{n_{v_{y_2}}}\Big), & \text{if } (\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}) \notin \mathfrak{E}_{E_2}. \end{cases}$$

Here, $n_{u_{x_2}}$ *and* $n_{v_{y_2}}$ *denote the counts of edges incident on vertices* $\mathfrak{u}_{x_2}$ *and* $\mathfrak{v}_{y_2}$ *in* $\mathfrak{G}_{G_2}$*.*

Definition 3.7. *The* ***Singular M-Lexicographic Product*** *of two Einstein fuzzy graphs* $\mathcal{G}_{G_1} = (\psi_{V_1}, \xi_{E_1})$ *on* $\mathfrak{G}_{G_1} = (\mathfrak{V}_{V_1}, \mathfrak{E}_{E_1})$ *and* $\mathcal{G}_{G_2} = (\psi_{V_2}, \xi_{E_2})$ *on* $\mathfrak{G}_{G_2} = (\mathfrak{V}_{V_2}, \mathfrak{E}_{E_2})$*, denoted by,*

$$\mathcal{G}_{G_1}[\mathcal{G}_{G_2}]_{S_M} = (\psi_{V_1} \circ_{S_M} \psi_{V_2}, \xi_{E_1} \circ_{S_M} \xi_{E_2}),$$

is defined as follows. Here, ψ_{V_i} *and* ξ_{E_i} *are fuzzy subsets of* $\mathfrak{V}_{V_i}$ *and* $\mathfrak{E}_{E_i}$*, respectively, for* $i \in \{1, 2\}$*:*

i. The vertex membership function is defined by:

$$(\psi_{V_1} \circ_{S_M} \psi_{V_2})(\mathfrak{u}_{x_1}, \mathfrak{u}_{x_2}) = \frac{\psi_{V_1}(\mathfrak{u}_{x_1}) \cdot \psi_{V_2}(\mathfrak{u}_{x_2})}{1 + \Big(1 - \psi_{V_1}(\mathfrak{u}_{x_1})\Big)\Big(1 - \psi_{V_2}(\mathfrak{u}_{x_2})\Big)},$$

for all $(\mathfrak{u}_{x_1}, \mathfrak{u}_{x_2}) \in \mathfrak{V}_{V_1} \times \mathfrak{V}_{V_2}$.

ii. The edge membership function is given by:

$$(\xi_{E_1} \circ_{S_M} \xi_{E_2})\Big((\mathfrak{u}_{x_1}, \mathfrak{u}_{x_2}), (\mathfrak{v}_{y_1}, \mathfrak{v}_{y_2})\Big) = \frac{\xi_{E_1}(\mathfrak{u}_{x_1}, \mathfrak{v}_{y_1}) \cdot \xi^M_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2})}{1 + \Big(1 - \xi_{E_1}(\mathfrak{u}_{x_1}, \mathfrak{v}_{y_1})\Big)\Big(1 - \xi^M_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2})\Big)},$$

for all $(\mathfrak{u}_{x_1}, \mathfrak{v}_{y_1}) \in \mathfrak{E}_{E_1}$ *and* $(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}) \in \mathfrak{V}_{V_2} \times \mathfrak{V}_{V_2}$ *such that* $(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}) \notin \mathfrak{E}_{E_2}$*, where*

$$\xi^M_{E_2}(\mathfrak{u}_{x_2}, \mathfrak{v}_{y_2}) = \mathfrak{T}_T\Big([\psi_{V_2}(\mathfrak{u}_{x_2})]^{n_{u_{x_2}}}, [\psi_{V_2}(\mathfrak{v}_{y_2})]^{n_{v_{y_2}}}\Big).$$

Example 3.5. *Refer to the two Einstein fuzzy graphs, $\mathcal{G}_{G1}$ and $\mathcal{G}_{G2}$, as presented in Example 3.1. The M-lexicographic product $\mathcal{G}_{G1}[\mathcal{G}_{G2}]_M$ is computed and illustrated in Table 1 and Figure 11. Additionally, the Singular M-lexicographic product $\mathcal{G}_{G1}[\mathcal{G}_{G2}]_{S_M}$ is depicted in Figure 12. The membership values of the edges are also obtained from Table 1.*

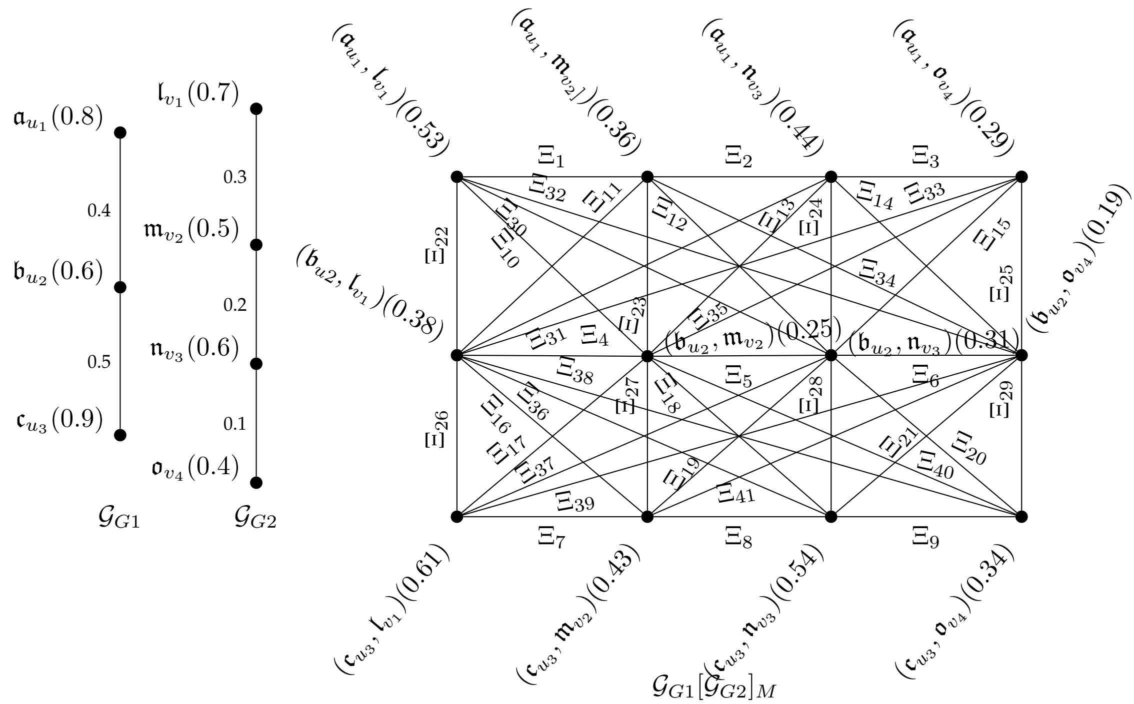

Figure 11: Einstein fuzzy graph: $\mathcal{G}_{G1}$, $\mathcal{G}_{G2}$ and $\mathcal{G}_{G1}[\mathcal{G}_{G2}]_M$.

Table 1: Membership values of the edges in $\mathcal{G}_{G1}[\mathcal{G}_{G2}]_M$.

Edges in $\mathcal{G}_{G1} \circ_M \mathcal{G}_{G2}$	Membership function $\xi_{E1} \circ_M \xi_{E2}$	Membership values	Representations
$((\mathfrak{a}_{u_1},\mathfrak{l}_{v_1}),(\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{l}_{v_1}),(\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}))$	0.1450	Ξ_1
$((\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}),(\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}),(\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}))$	0.0940	Ξ_2
$((\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}),(\mathfrak{a}_{u_1},\mathfrak{o}_{v_4}))$	$(\xi_{E1}\circ_M\xi_2)((\mathfrak{a}_{u_1},\mathfrak{n}_{v_1}),(\mathfrak{a}_{u_1},\mathfrak{o}_{v_4}))$	0.0457	Ξ_3
$((\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}),(\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}),(\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}))$	0.0169	Ξ_4
$((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}))$	0.0110	Ξ_5
$((\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}),(\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}),(\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}))$	0.0051	Ξ_6
$((\mathfrak{c}_{u_3},\mathfrak{l}_{v_1}),(\mathfrak{c}_{u_3},\mathfrak{m}_{v_2}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{c}_{u_3},\mathfrak{l}_{v_1}),(\mathfrak{c}_{u_3},\mathfrak{m}_{v_2}))$	0.2110	Ξ_7
$((\mathfrak{c}_{u_3},\mathfrak{m}_{v_2}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{c}_{u_3},\mathfrak{m}_{v_3}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	0.1380	Ξ_8
$((\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}),(\mathfrak{c}_{u_3},\mathfrak{o}_{v_4}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}),(\mathfrak{c}_{u_3},\mathfrak{o}_{v_4}))$	0.0680	Ξ_9
$((\mathfrak{a}_{u_1},\mathfrak{l}_{v_1}),(\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{l}_{v_1}),(\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}))$	0.0850	Ξ_{10}
$((\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}),(\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}),(\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}))$	0.0850	Ξ_{11}
$((\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}),(\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}),(\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}))$	0.0540	Ξ_{12}
$((\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}),(\mathfrak{b}_{u_2},\mathfrak{m}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}),(\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}))$	0.0540	Ξ_{13}
$((\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}),(\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}),(\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}))$	0.0260	Ξ_{14}
$((\mathfrak{a}_{u_1},\mathfrak{o}_{v_4}),(\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{o}_{v_4}),(\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}))$	0.0260	Ξ_{15}
$((\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}),(\mathfrak{c}_{u_3},\mathfrak{m}_{v_2}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}),(\mathfrak{c}_{v_3},\mathfrak{m}_{v_2}))$	0.1110	Ξ_{16}
$((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{c}_{u_3},\mathfrak{l}_{v_1}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{c}_{v_3},\mathfrak{l}_{v_1}))$	0.1110	Ξ_{17}
$((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{c}u3,\mathfrak{n}_{v_3}))$	0.0710	Ξ_{18}
$((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	0.0710	Ξ_{19}
$((\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}),(\mathfrak{c}_{u_3},\mathfrak{o}_{v_4}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}),(\mathfrak{c}_{u_3},\mathfrak{o}_{v_4}))$	0.0350	Ξ_{20}
$((\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	0.0350	Ξ_{21}
$((\mathfrak{a}_{u_1},\mathfrak{l}_{v_1}),(\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{l}_{v_1}),(\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}))$	0.1350	Ξ_{22}
$((\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}),(\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}),(\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}))$	0.0102	Ξ_{23}
$((\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}),(\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}),(\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}))$	0.0238	Ξ_{24}
$((\mathfrak{a}_{u_1},\mathfrak{o}_{v_4}),(\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{o}_{v_4}),(\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}))$	0.0308	Ξ_{25}
$((\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}),(\mathfrak{c}_{u_3},\mathfrak{l}_{v_1}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}),(\mathfrak{c}_{u_3},\mathfrak{l}_{v_1}))$	0.1762	Ξ_{26}
$((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{c}_{u_3},\mathfrak{m}_{v_2}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{c}_{u_3},\mathfrak{m}_{v_2}))$	0.0135	Ξ_{27}
$((\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	0.0316	Ξ_{28}
$((\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}),(\mathfrak{c}_{u_3},\mathfrak{o}_{v_4}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}),(\mathfrak{c}_{u_3},\mathfrak{o}_{v_4}))$	0.0408	Ξ_{29}
$((\mathfrak{a}_{u_1},\mathfrak{l}_{v_1}),(\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{l}_{v_1}),(\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}))$	0.0573	Ξ_{30}
$((\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}),(\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{n}_{v_3}),(\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}))$	0.0573	Ξ_{31}
$((\mathfrak{a}_{u_1},\mathfrak{l}_{v_1}),(\mathfrak{b}_{u_2},\mathfrak{o}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{l}_{v_1}),(\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}))$	0.0650	Ξ_{32}
$((\mathfrak{a}_{u_1},\mathfrak{o}_{v_4}),(\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{o}_{v_4}),(\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}))$	0.0650	Ξ_{33}
$((\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}),(\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{m}_{v_2}),(\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}))$	0.0177	Ξ_{34}
$((\mathfrak{a}_{u_1},\mathfrak{o}_{v_4}),(\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{a}_{u_1},\mathfrak{o}_{v_4}),(\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}))$	0.0177	Ξ_{35}
$((\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}),(\mathfrak{c}_{u_3},\mathfrak{n}_{v_3}))$	0.0757	Ξ_{36}
$((\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}),(\mathfrak{c}_{u_2},\mathfrak{l}_{v_1}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{n}_{v_3}),(\mathfrak{c}_{u_3},\mathfrak{l}_{v_1}))$	0.0757	Ξ_{37}
$((\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}),(\mathfrak{c}_{u_3},\mathfrak{o}_{v_4}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{l}_{v_1}),(\mathfrak{c}_{u_3},\mathfrak{o}_{v_4}))$	0.0858	Ξ_{38}
$((\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}),(\mathfrak{c}_{u_3},\mathfrak{l}_{v_1}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}),(\mathfrak{c}_{u_3},\mathfrak{l}_{v_1}))$	0.0858	Ξ_{39}
$((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{c}_{u_3},\mathfrak{o}_{v_4}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{m}_{v_2}),(\mathfrak{c}_{u_3},\mathfrak{o}_{v_4}))$	0.0235	Ξ_{40}
$((\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}),(\mathfrak{c}_{u_3},\mathfrak{m}_{v_2}))$	$(\xi_{E1}\circ_M\xi_{E2})((\mathfrak{b}_{u_2},\mathfrak{o}_{v_4}),(\mathfrak{c}_{u_3},\mathfrak{m}_{v_2}))$	0.0235	Ξ_{41}

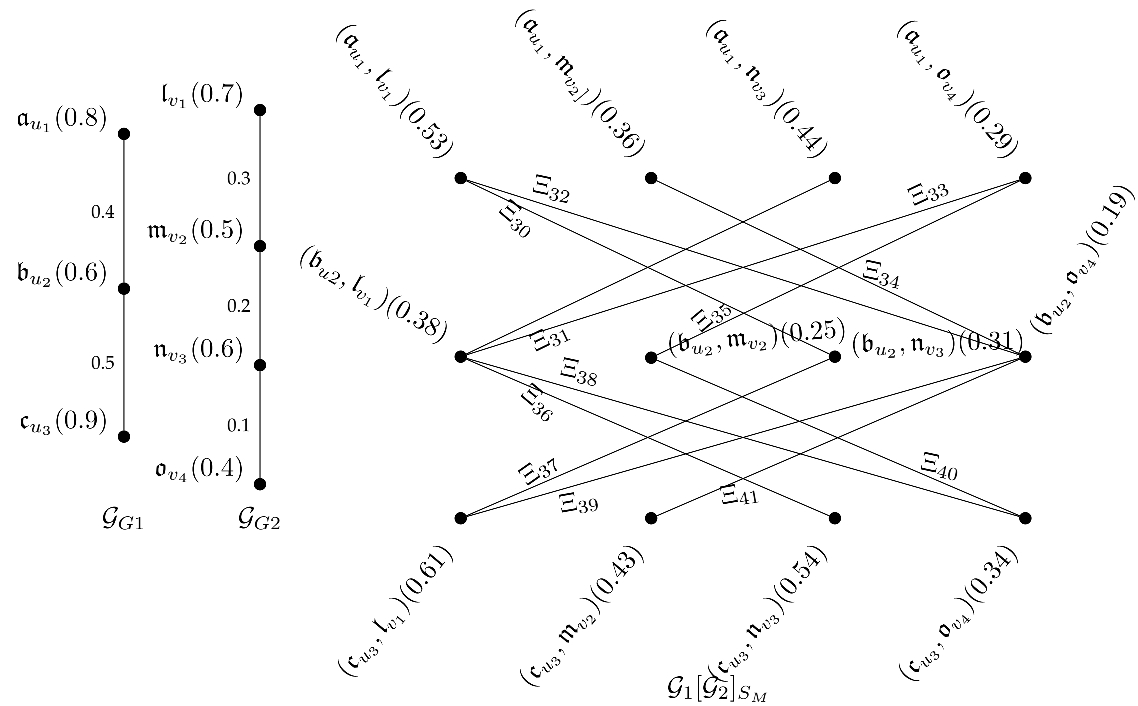

Figure 12: Einstein fuzzy graph: $\mathcal{G}_{G1}$, $\mathcal{G}_{G2}$ and $\mathcal{G}_{G_1}[\mathcal{G}_{G_2}]_{S_M}$.

Theorem 3.4. *The* ***M-lexicographic product*** $\mathcal{G}_{G_1}[\mathcal{G}_{G_2}]_M$ *of* $\mathcal{G}_{G_1}$ *and* $\mathcal{G}_{G_2}$ *is the Einstein fuzzy graph corresponding to the lexicographic product* $\mathfrak{G}_{G_1}[\mathfrak{G}_{G_2}]$*. Here,* $\mathcal{G}_{G_1}$ *and* $\mathcal{G}_{G_2}$ *are the Einstein fuzzy graphs of the graphs* $\mathfrak{G}_{G_1}$ *and* $\mathfrak{G}_{G_2}$*, respectively.*

Proof. It is enough to prove that,

$$(\xi_{E1}\circ_M \xi_{E2})\Big((\mathfrak{u}_{x1},\mathfrak{u}_{x2})(\mathfrak{v}_{y1},\mathfrak{v}_{y2})\Big) \le \frac{(\psi_{V1}\circ_M\psi_{V2})(\mathfrak{u}_{x1},\mathfrak{u}_{x2})(\psi_{V1}\circ_M\psi_{V2})(\mathfrak{v}_{y1},\mathfrak{v}_{y2})}{1+\Big(\Big[1-(\psi_{V1}\circ_M\psi_{V2})(\mathfrak{u}_{x1},\mathfrak{u}_{x2})\Big]\Big[1-(\psi_{V1}\circ\psi_{V2})(\mathfrak{v}_{y1},\mathfrak{v}_{y2})\Big]\Big)},$$

for every $(\mathfrak{u}_{x1},\mathfrak{v}_{y1})\in\mathfrak{E}_{E1}$, for every $(\mathfrak{u}_{x2},\mathfrak{v}_{y2})\notin\mathfrak{E}_{E2}$.

Consider $(\mathfrak{u}_{x1},\mathfrak{v}_{y1})\in\mathfrak{E}_{E1}$ and $(\mathfrak{u}_{x2},\mathfrak{v}_{y2})\notin\mathfrak{E}_{E2}$,

$$\begin{aligned}(\xi_{E1}\circ_M \xi_{E2})\Big((\mathfrak{u}_{x1},\mathfrak{u}_{x2})(\mathfrak{v}_{y1},\mathfrak{v}_{y2})\Big) &= \frac{\xi_{E1}(\mathfrak{u}_{x1},\mathfrak{v}_{y1})\xi^M_{E2}(\mathfrak{u}_{x2},\mathfrak{v}_{y2})}{1+\Big(1-\xi_{E1}(\mathfrak{u}_{x1},\mathfrak{v}_{y1})\Big)\Big(1-\xi^M_{E2}(\mathfrak{u}_{x2},\mathfrak{v}_{y2})\Big)}\\ &= \frac{\xi_{E1}(\mathfrak{u}_{x1},\mathfrak{v}_{y1})\mathfrak{T}_T([\psi_{V2}(\mathfrak{u}_{x2})]^{n_{u_2}},[\psi_{V2}(\mathfrak{v}_{y2})]^{n_{v_2}})}{1+\Big(1-\xi_{E1}(\mathfrak{u}_{x1},\mathfrak{v}_{y1})\Big)\Big(1-\mathfrak{T}_T\Big([\psi_{V2}(\mathfrak{u}_{x2})]^{n_{u_2}},[\psi_{V2}(\mathfrak{v}_{y2})]^{n_{v_2}}\Big)\Big)}.\end{aligned}$$

We have

$$\begin{aligned}\xi_{E1}(\mathfrak{u}_{x1},\mathfrak{v}_{y1}) &\le \mathfrak{T}_T\Big(\psi_{V1}(\mathfrak{u}_{x1}),\psi_{V1}(\mathfrak{v}_{y1})\Big),\\ \Big[\psi_{V2}(\mathfrak{u}_{x2})\Big]^{n_{\mathfrak{u}_{x2}}} &\le \psi_{V2}(\mathfrak{u}_{x2}),\\ \Big[\psi_{V2}(\mathfrak{v}_{y2})\Big]^{n_{\mathfrak{v}_{y2}}} &\le \psi_{V2}(\mathfrak{v}_{y2}).\end{aligned}$$

Hence,

$$\mathfrak{T}_T\Big([\psi_{V2}(\mathbb{u}_{x2})]^{n_{\mathbb{u}_{x2}}}, [\psi_{V2}(\mathbb{v}_{y2})]^{n_{\mathbb{v}_{y2}}}\Big) \leq \mathfrak{T}_T\Big(\psi_{V2}(\mathbb{u}_{x2}), \psi_{V2}(\mathbb{v}_{y2})\Big),$$

i.e.,

$$\begin{aligned}(\xi_{E1} \circ_M \xi_{E2})\Big((\mathfrak{u}_{x1}, \mathbb{u}_{x2})(\mathbb{v}_{y1}, \mathbb{v}_{y2})\Big) &= \frac{\xi_{E1}(\mathbb{u}_{x1}, \mathbb{v}_{y1})\xi^M_{E2}(\mathbb{u}_{x2}, \mathbb{v}_{y2})}{1 + \Big(1 - \xi_{E1}(\mathfrak{u}_{x1}, \mathfrak{v}_{y1})\Big)\Big(1 - \psi^M_{E2}(\mathfrak{u}_{x2}, \mathbb{v}_{y2})\Big)} \\ &= \mathfrak{T}_T\Big(\xi_{E1}(\mathbb{u}_{x1}, \mathbb{v}_{y1}), \xi^M_{E2}(\mathfrak{u}_{x2}, \mathfrak{v}_{y2})\Big) \\ &\leq \mathfrak{T}_T\Big(\mathfrak{T}_T\Big(\psi_{V1}(\mathbb{u}_{x1}), \psi_{V1}(\mathbb{v}_{y1})\Big), \mathfrak{T}_T\Big(\psi_{V2}(\mathbb{u}_{x2}), \psi_{V2}(\mathbb{v}_{y2})\Big)\Big).\end{aligned}$$

$$\begin{aligned}&(\xi_{E1} \circ_M \xi_{E2})\Big((\mathbb{u}_{x1}, \mathfrak{u}_{x2})(\mathbb{v}_{y1}, \mathbb{v}_{y2})\Big) \\ &\leq \mathfrak{T}_T\Big(\mathfrak{T}_T\Big(\psi_{V1}(\mathbb{u}_{x1}), \psi_{V1}(\mathbb{v}_{x1})\Big), \mathfrak{T}_T\Big(\psi_{V2}(\mathbb{u}_{x2}), \psi_{V2}(\mathbb{v}_{y2})\Big)\Big) \\ &\leq \frac{\mathfrak{T}_T\Big(\psi_{V1}(\mathbb{u}_{x1}), \psi_{V1}(\mathbb{v}_{y1})\Big)\mathfrak{T}_T\Big(\psi_{V2}(\mathbb{u}_{x2}), \psi_{V2}(\mathbb{v}_{y2})\Big)}{1 + \Big[1 - \mathfrak{T}_T\Big(\psi_{V1}(\mathfrak{u}_{x1}), \psi_{V1}(\mathfrak{v}_{y1})\Big)\Big]\Big[1 - \mathfrak{T}_T\Big(\psi_{V2}(\mathfrak{u}_{x2}), \psi_{V2}(\mathfrak{v}_{y2})\Big)\Big]} \\ &\leq \frac{\frac{\psi_{V1}(\mathbb{u}_{x1})\psi_{V1}(\mathbb{v}_{y1})}{1 + [(1 - \psi_{V1}(\mathbb{u}_{x1}))(1 - \psi_{V1}(\mathbb{v}_{y1}))]}\frac{\psi_{V2}(\mathbb{u}_{x2})\psi_{V2}(\mathbb{v}_{y2})}{1 + [(1 - \psi_{V2}(\mathbb{u}_{x2}))(1 - \psi_{V2}(\mathbb{v}_{y2}))]}}{1 + \Big[\Big(1 - \frac{\psi_{V1}(\mathbb{u}_{x1})\psi_{V1}(\mathfrak{v}_{y1})}{1 + [(1 - \psi_{V1}(\mathbb{u}_{x1}))(1 - \psi_{V1}(\mathbb{v}_{y1}))]}\Big)\Big(1 - \frac{\psi_{V2}(\mathbb{u}_{x2})\psi_{V2}(\mathbb{v}_{y2})}{1 + [(1 - \psi_{V2}(\mathbb{u}_{x2}))(1 - \psi_{V2}(\mathfrak{v}_{y2}))]}\Big)\Big]} \\ &\leq \frac{\frac{\psi_{V1}(\mathbb{u}_{x1})\psi_{V2}(\mathbb{u}_{x2})}{1 + [(1 - \psi_{V1}(\mathbb{u}_{x1}))(1 - \psi_{V2}(\mathbb{u}_{x2}))]}\frac{\psi_{V1}(\mathbb{v}_{y1})\psi_{V2}(\mathbb{v}_{y2})}{1 + [(1 - \psi_{V1}(\mathbb{v}_{y1}))(1 - \psi_{V2}(\mathbb{v}_{y2}))]}}{1 + \Big[\Big(1 - \frac{\psi_{V1}(\mathbb{u}_{x1})\psi_{V2}(\mathbb{u}_{x2})}{1 + \big[(1 - \psi_{V1}(\mathbb{u}_{x1}))(1 - \psi_{V2}(\mathbb{u}_{x2}))\big]}\Big)\Big(1 - \frac{\psi_{V1}(\mathbb{v}_{y1})\psi_{V2}(\mathbb{v}_{y2})}{1 + \big[(1 - \psi_{V1}(\mathbb{v}_{y1}))(1 - \psi_{V2}(\mathbb{v}_{y2}))\big]}\Big)\Big]} \\ &\leq \frac{(\psi_{E1} \circ_M \psi_{E2})(\mathbb{u}_{x1}, \mathbb{u}_{x2})(\psi_{E1} \circ_M \psi_{E2})(\mathbb{v}_{y1}, \mathbb{v}_{y2})}{1 + \Big(\Big[1 - (\psi_{V1} \circ_M \psi_{V2})(\mathbb{u}_{x1}, \mathbb{u}_{x2})\Big]\Big[1 - (\psi_{V1} \circ \psi_{V2})(\mathbb{v}_{y1}, \mathbb{v}_{y2})\Big]\Big)}.\end{aligned}$$

$$(\xi_{E1} \circ_M \xi_{E2})\Big((\mathbb{u}_{x1}, \mathbb{u}_{x2})(\mathbb{v}_{y1}, \mathfrak{v}_{y2})\Big) \leq \mathfrak{T}\Big((\psi_{V1} \circ_M \psi_{v2})(\mathbb{u}_{x1}, \mathbb{u}_{x2})(\psi_{V1} \circ_M \psi_{V2})(\mathbb{v}_{y1}, \mathbb{v}_{y2})\Big).$$

Therefore, $\mathcal{G}_{G1}[\mathcal{G}_{G2}]_M$ is an Einstein fuzzy graph of $\mathfrak{G}_{G1}[\mathfrak{G}_{G2}]$. □

Corollary 3.2. *Let $\mathcal{G}_{G_1}$ and $\mathcal{G}_{G_2}$ be the Einstein fuzzy graphs corresponding to the crisp graphs $\mathfrak{G}_{G_1}$ and $\mathfrak{G}_{G_2}$, respectively. Then, the* ***Singular M-lexicographic product*** *$\mathcal{G}_{G_1}[\mathcal{G}_{G_2}]_{S_M}$ is also an Einstein fuzzy graph.*

Proof. The result follows immediately from the above theorem. □

Proposition 3.2 (Properties of the M-Lexicographic Product)**.** *The M-Lexicographic product $\mathcal{G}_{G_1}[\mathcal{G}_{G_2}]_M$ possesses the following properties:*

- *It includes all the edges of the direct product $\mathcal{G}_{G_1} \times \mathcal{G}_{G_2}$, preserving the corresponding membership values.*
- *It includes all the edges of the Quasi I_M Cartesian product and Quasi II_M Cartesian product, also preserving the same membership values.*

- *It includes all the edges of the M-Cartesian product, M-Semi Strong product and M-Strong product, also preserving the same membership values.*
- *It can be regarded as the union of the edge sets of the M-Strong product and the Singular M-Lexicographic product and it includes the same vertices with the same membership values.*

4 Applications of Einstein and M-Einstein Fuzzy Graphs

Here, we explore the applications of Einstein and M-Einstein fuzzy graphs in the medical field. A person's health without communicable diseases, as well as the impact of such diseases on health, is illustrated using these fuzzy graphs and their associated products.

Consider three persons $\{P_1, P_2, P_3\}$. The vertices of the graph represent the health states of these individuals, with membership values between 0 and 1. These membership values indicate the degree to which each person is healthy (with 0 meaning completely unhealthy and 1 meaning completely healthy). We can represent this situation using an Einstein fuzzy graph, where the health states of individuals and their interactions are modeled with fuzzy membership values, as shown in the Figure 13 below.

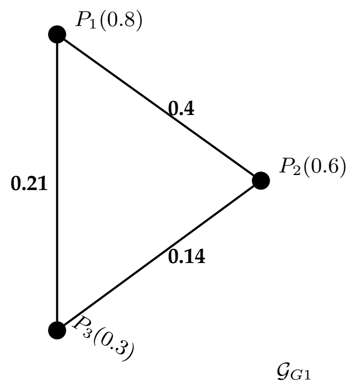

Figure 13: Analyzing health states and interactions of individuals with Einstein fuzzy graphs: $\mathcal{G}_1$.

Consider another Einstein Fuzzy Graph with the following conditions: Two symptoms $\{S_1, S_2\}$of the communicable disease D_1 are represented by the vertices of the graph. The membership value of each vertex corresponds to the health condition of individuals after experiencing the symptoms or effects ofS_1 from D_1.

For example, if the symptom is body pain, the membership value of the vertex represents how the body pain affects or worsens the individual's condition. The membership value of the edge between the two symptoms represents the strength of their combined effect on the person's health as Figure 14.

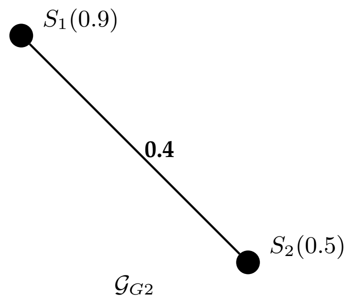

Figure 14: Analyzing symptom interactions in Einstein fuzzy graphs: $\mathcal{G}_2$.

Consider the Cartesian product of two Einstein fuzzy graphs, $\mathcal{G}_1$ and $\mathcal{G}_2$, as shown in Figure 15.

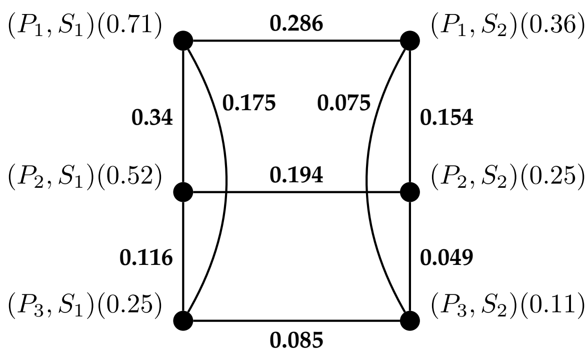

Figure 15: Fuzzy graph of health states and symptom effects: $\mathcal{G}_{G1} \times \mathcal{G}_{G2}$.

The fuzzy graph $\mathcal{G}_{G1} \times \mathcal{G}_{G2}$ not an Einstein fuzzy graph as Figure 16.

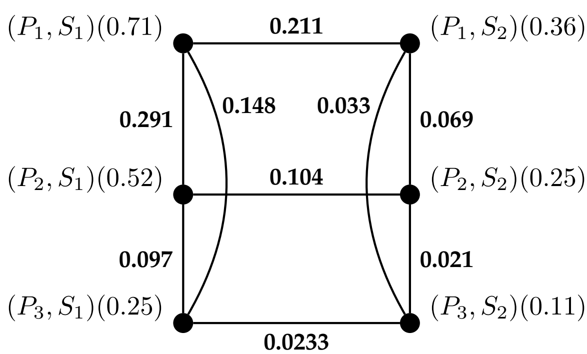

Figure 16: Einstein fuzzy graph of health states and symptom effects: $\mathcal{G}_{G1} \times_M \mathcal{G}_{G2}$.

The Cartesian product of Einstein fuzzy graphs is not necessarily an Einstein fuzzy graph, but the modified product of Einstein fuzzy graphs is an Einstein fuzzy graph.

From the Cartesian product of two fuzzy graphs, we can derive the health conditions of three people affected by disease D_1. Each vertex provides information about how the health condition of persons P_1, P_2 and P_3 evolves, as reflected by the membership values of the vertices. In the modified product, the membership values of the edges are evaluated under the Einstein t-norm. The vertex (P_1, S_1) with a membership value of 0.71 represents the health condition of person 1 after being affected by symptom S_1. The membership value of the edge between (P_1, S_1) and (P_1, S_2) represents the health condition of person 1 after being affected by both symptoms S_1 and S_2 or after the impact of disease D_1. The membership value of the edge between (P_1, S_1) and (P_2, S_1) reflects how the relationship between person 1 and person 2 develops after the effect of symptom S_1.

In the modified products of Einstein fuzzy graphs, the membership value of an edge is lower than the corresponding edge values in the non-modified products. This characteristic is advantageous for our example. Some edges represent the influence of the communicable disease D1 on two symptoms. In such cases, identifying or selecting a medicine that effectively treats disease D1 becomes more feasible. The reduced membership values help highlight the most applicable treatment option for the disease. Similarly, some edges represent the relationship between two individuals at the time of infection. This information can support decision-making regarding enhanced protective measures for those individuals during the infection period.

In this section, we focus solely on the application of products of Einstein fuzzy graphs in the context of Cartesian products within the medical field. However, these applications can be extended to all types of graph products discussed in this paper. Furthermore, various types of products presented in this work can find potential applications in diverse domains such as agriculture, data analytics, banking and more.

5 Conclusions

This study introduces a novel approach to analyzing modified products within the framework of Einstein fuzzy graphs. In general, operations such as the Quasi-I Cartesian product, Quasi-II Cartesian product, Cartesian product, Semi-strong product, Symmetric composition and Composition of two Einstein fuzzy graphs do not necessarily result in another Einstein fuzzy graph. To address this limitation, the study proposes modifications to these product operations to ensure that the resulting graphs remain within the class of Einstein fuzzy graphs. The properties discussed provide a comprehensive overview of these modified operations. For future research, the concept of product operations can be extended to the intuitionistic fuzzy graph environment by incorporating the Einstein transformation. Furthermore, exploring potential applications of these modified operations in various real-life domains presents a promising direction for future work.

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Conflicts of Interest The authors declare no conflict of interest.

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